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August. 1. 1853.

THE

MATHEMATICAL QUESTIONS

OF THE

SENATE-HOUSE EXAMINATION PAPERS,

1838 to 1849

INCLUSIVE,

ARRANGED IN ORDER ACCORDING TO SUBJECTS,

BY THE REV. A. H. FROST, M. A.

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PREFACE.

THE design of the Compiler of these questions has been to present in a form convenient for use or reference the Senate-house Papers from the years 1838 to 1849 inclusive.

The general plan which has been adopted will be at once seen from an inspection of the work itself. The "Problem papers" of each year in succession, have been arranged in order according to subjects, and form Part I.: The remaining Papers consist of questions which have appeared in published works; these are commonly called "Bookwork Questions," and to these are frequently annexed illustrations of the principle contained in them, which are usually termed "Riders."

These (the riders) have been carefully separated from the Bookwork for the use of Students, who require an easier set of examples, than is furnished in the Problem Papers.

The Bookwork questions form Part II.; but in consequence of the changes in the University System which have been gradually adopted during the years 1848 and 1849, the compiler has thought that the publication of the whole, from 1838, would needlessly increase the bulk without adding to the usefulness of the work. The years 1845—9 have therefore alone been introduced.

The Riders form Part III. Where any question has belonged to more than one subject, it has either been classed under that head to which the stress of the problem mainly applied, or it has been repeated in those subjects by whose principles it might be solved.

In order to understand the changes recently made, and which are still in progress, reference should be made to the Schedules and regulations issued by the "board of Mathematical Studies," and especially that established by Grace, Oct. 31, 1848, and the last annual report.

For the convenience of those who may not have access to these, it may be useful to state generally, that at present the examination, during the first three days, is to consist of the Elementary parts only of the following subjects:—Euclid, Arithmetic, Algebra, Plane Trigonometry, Conic Sections, (treated Geometrically), Statics and Dynamics (treated without the Differential Calculus), Sections, 1, 2, 3 of Newton's Principia, Hydrostatics, Optics and Astronomy, as also Examples arising directly out of the propositions. The questions which have been proposed during these three days of 1848 and 1849 are distinguished by an asterisk.

MATHEMATICAL QUESTIONS.

PART I.

Those Problems with an asterisk were proposed in the first three days of 1848, & 1849.

EUCLID.

1. If any two consecutive sides of a hexagon inscribed in a circle be respectively parallel to their opposite sides, the remaining sides are parallel to each other. 1838

2. $ABCD$ is a rectangular parallelogram, of which A, C are opposite angles, E any point in BC, F any point in CD. Prove that 2 area of triangle AEF together with the rectangle BE, DF is equal to the parallelogram AC.

A postman starting from A_1 has to deliver letters at A_2, A_3..., A_n, forming with his own station successively the n sides of a polygon; shew that for the shortest route his first delivery must be either at A_2, or A_n. 1839

Supposing the distances A_1A_2, A_2A_3,...$A_{n-1}A_n$ all equal, when there are only three places of delivery, or if there be more than three and none of the angles at A_3, A_4,..., A_{n-1}, acute, determine the complete path.

Two circles intersect in A and B; AD, AD' are diameters; AC, AC' are chords, each of which touches the circle of which it is not a chord; the line AEE' bisects the angle DAD' and cuts the circles in E and E'; then the common tangent to the circles is a mean proportional between the chords DE, $D'E'$; and their common chord (AB) is a mean proportional between the chords BC, BC'. 1840

If two equal circles cut each other, and if through one of the points of intersection a line be drawn terminated by the 1842

circles, the lines joining its extremities with the other point of intersection are equal.

1843 1. In a quadrant of a circle two semicircles are inscribed, their diameters coinciding with the radii of the quadrant; shew that the area common to the two semicircles is equal to that portion of the area of the quadrant which is outside the semicircles.

2. From a given point outside two given circles, which do not meet, to draw a straight line, such, that the portions of it intercepted by each circle shall be respectively proportional to their radii.

3. Bisect a right-angled triangle by a line perpendicular to its hypothenuse.

4. A, B, C, A', B', C', are points on the circumference of a circle; if the lines AB, AC be respectively parallel to $A'B'$, $A'C'$, shew that BC' is parallel to $B'C$.

1844 Describe three circles touching each other and having their centres at three given points. In how many different ways may this be done?

1845 1. The squares of the sides of a quadrilateral figure are greater than the squares of its diagonals by four times the square of the line joining the middle points of the diagonals.

2. Divide a circle into two parts such that the angle contained in one segment shall equal twice the angle contained in the other.

1846 Two equal circles intersect at right angles, and with the points of intersection as centres two arcs are drawn touching the circles, so as to form an oval; shew that the space common to the two circles is equal to each of the spaces exterior to both.

1847 1. A straight line and two circles are given; find the point in the straight line from which two tangents drawn to the circles shall be equal.

2. In the figure of Euclid, Book I. Prop. 35, (*Parallelograms on the same base and between the same parallels are equal,*)

if two diagonals be drawn to the two parallelograms respectively, one from each extremity of the base, and the intersection of the diagonals be joined with the intersection of the sides (or sides produced) in the figure, prove that the joining line will bisect the base.

3. AB is a common chord of the segments ACB, $ADEB$ of two circles, and through C, any point in ACB, are drawn the straight lines ACE, BCD; prove that the arc DE is invariable.

1. *If the hypotenuse AB of a right-angled triangle ABC 1848
be bisected in D, and EDF drawn perpendicular to AB, and DE, DF cut off each equal to DA, and CE, CF joined, prove that the last two lines will bisect the angle at C and its supplement respectively.

2. *A, B, C are three given points in the circumference of a given circle; find a point P such that if AP, BP, CP meet the circumference in D, E, F the arcs DE, EF, may be equal to given arcs.

3. AB, CD are any two chords of a circle passing through a fixed point, O, EF any chord parallel to AB; join CE, DF meeting AB in the points G and H, and DE, CF meeting AB in the points K and L; shew that the rectangle $OG \cdot OH = OK \cdot OL$.

4. In a given circle inscribe a rectangle equal to a given rectilineal figure.

5. Through a given point describe a circle which shall touch a given circle and intersect another given circle in a chord passing through a given point.

1. *Through a point C in the circumference of a circle, two 1849
straight lines ACB, DCE, are drawn, cutting the circle in B and E; prove that the straight line which bisects the angles ACE, DCB meets the circle in a point equidistant from B and E.

2. *Two circles intersect in A and B. At A, the tangents AC, AD are drawn to each circle and terminated by the circumference of the other. If BC, BD be joined, shew that AB, or AB produced if necessary, bisects the angle CBD.

3. * Draw a line to touch one given circle, so that the part of it contained by another given circle shall be equal to a given straight line not greater than the diameter of this latter circle.

4. * A quadrilateral figure possesses the following property; any point being taken and four triangles formed by joining this point with the angular points of the figure, the centres of gravity of these triangles lie in the circumference of a circle; prove that the diagonals of the quadrilateral are at right angles to each other.

5. Three circles are described, each of which touches one side of a triangle ABC, and the other two sides produced. If D be the point of contact of the side BC, E that of AC, and F that of AB, shew that $AE = BD$, $BF = CE$ and $CD = AF$.

ARITHMETIC.

1839 If the price in shillings of a cwt. of goods be multiplied by 3 and divided by 7, the result is the value in farthings of a pound weight.

1844 The difference between a certain multiple of 10 and the sum of its digits is 99: find it.

1847 A piece of paper a mile long is rolled into a solid cylinder; find approximately the diameter of the cylinder, supposing 240 leaves of such paper to have a thickness of one inch.

ALGEBRA.

1838 1. A SCALE of French inches is placed by the side of a scale of English inches so that the first divisions of the two scales coincide; find the divisions of the English scale that coincide most nearly with those of the French scale. The length of each scale is 10 feet, and 100 French inches = 106,577 English inches.

2. If the values of x, y, x', y' be all possible, and

$$1 + xx' + yy' = \sqrt{1 + x^2 + y^2} \cdot \sqrt{1 + x'^2 + y'^2},$$

shew that $x = x'$ and $y = y'$.

3. Find the x^{th} term and the sum of x terms of the series 1, 2, 3, 8, 13, 30....

4. Prove that the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges without assuming that it is the series for $-\log_e(1-x)$ when $x = 1$.

5. If μ_n denote the n^{th} quotient of a continued fraction, p_n, q_n the numerator and denominator of the corresponding converging fractions, shew that, the fraction being greater than unity,

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_{n-1}}{q_{n-1}} = (-1)^n \cdot \frac{\mu_{n+1}}{q_{n+1} \cdot q_{n-1}};$$

and that each of the fractions

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}}, \quad \frac{2p_n + p_{n-1}}{2q_n + q_{n-1}}, \ldots \frac{(\mu_{n+1} - 1) \cdot p_n + p_{n-1}}{(\mu_{n+1} - 1) \cdot q_n + q_{n-1}},$$

approximates to the value of the continued fraction more nearly than any fraction having a smaller denominator.

1. Prove the following Equalities: 1839

$$\frac{d^m (a-b)(b-c) + b^m (a-d)(c-d)}{c^m (a-b)(a-d) + a^m (b-c)(c-d)} = \frac{b-d}{a-c} \text{ where } m = 0, 1, \text{or } 2.$$

2. Find the sum of (n) terms of the series

$$2 \cdot 1^2 + 3 \cdot 2^2 + 4 \cdot 3^2 + \ldots$$

3. Shew that the integral part of $(3 + \sqrt{5})^x$ is odd for all positive integer values of x; and give a rough calculation to prove that the integral part of $(\sqrt{951} + \sqrt{890})^{2x}$ is even for the first 64 values of x, and odd when x is any integer from 65 to 101.

4. Shew that, if A be an indefinite continued fraction in which all the terms from the first recur in order, and B a similar fraction composed of the same terms recurring in the opposite order, no equation can have A for a root without having $\frac{-1}{B}$ for another root.

5. If one number (A) have exactly as many places of figures as another (B), and also have more than the first half of its figures identical with the corresponding figures in (B);

the difference between $\sqrt[n]{A}$ and $\sqrt[n]{B}$ must be less than $\frac{1}{n}$, if (n) be any whole number not less than 2.

6. The number of balls contained in a truncated pile, of which the top and bottom are rectangular, is

$$\frac{p}{6} \cdot \{2p^2 + 3\,(m + n - 1)\,.\,p + 6mn - 3m - 3n + 1\},$$

where (m) and (n) represent the numbers of the balls in the two sides of the top, and (p) the number of the balls in each of the slanting edges.

7. Each of two dice has one face wrongly marked; the marks which ought to have been upon the two faces although unknown in numerical value are known to have been interchanged; shew that the chance of throwing 7 at one throw with both dice is $\frac{5}{28}$.

8. A person puts his hand into a bag containing 12 balls, draws out a certain number at random, and transfers them without examination to a second empty bag; he then puts his hand into this second bag, and draws out a certain number in the same way as before: shew that the odds in favour of his drawing out an odd number from the second bag are nearly as 341 to 340.

9. Four white balls and two red ones are placed promiscuously in a bag; two balls are then drawn; the remainder are then taken out of the bag and placed in one or more of the six pockets of a billiard table: find the chance that a person going to a single pocket will draw a red ball, supposing him not allowed to draw more than one.

10. The odds against m horses are n_1 to 1, n_2 to 1,. .,n_m to 1, shew that except in the particular case in which the sum of the reciprocals of $n_1 + 1$, $n_2 + 1$, &c. is unity, a person may so arrange his bets as to win a given sum whichever horse be successful; and that he must bet against or back every horse according as the above sum is greater or less than unity.

Taking the odds in the St. Leger 1838 against the horses

annexed to them as follows; 7 to 4 Don John, 9 to 4 Ion, 9 to 2 Lanercost, 9 to 1 Saintfoin, 15 to 1 Cobham, 20 to 1 Alzira, 33 to 1 Hydra; Arrange the bets, 1st so as to win £378. 10*s.* in any case, 2ndly so as to win £378. 10*s.* if either Don John or Lanercost be first, otherwise to be even.

11. If a person, having an unlimited number of similar regular polyhedrons, each of which has (m) faces, whilst each face has (n) sides, mark the different faces of each polyhedron with the numbers 1, 2, 3,. .($m-1$), m; it may happen that the first $\frac{1 \cdot 2 \cdot 3 \ldots (m-2) \cdot (m-1)}{n}$ polyhedrons will be so marked that the relative situations of the numbers shall not be the same upon any two of them.

12. A pack of np cards is dealt regularly round to p persons with their faces uppermost, every card dealt to each person being placed upon that previously dealt to him; the hands are then taken up, turned so as to have their backs uppermost, and placed upon one another; that hand which contains a particular card A being always placed below r other hands: the cards are then dealt again, the hands taken up, turned, and placed upon one another as before, and so on; shew that if m and q be the whole numbers next greater than $\frac{\log 2np}{\log p}$ and $\frac{rnp}{p-1}$ respectively, the card A will at the end of the m^{th} and every succeeding operation occupy the q^{th} place, or be restricted to the q^{th} and $q-1^{\text{th}}$ places from the top, according as rn is indivisible or divisible by $p-1$.

1. A fraction whose denominator is less than ten, when 1840
reduced to the decimal form, cannot contain the figure 9 in its decimal part. Are any other digits excluded in particular cases?

2. The advance of the hour-hand of a watch beyond the minute-hand is measured by $15\frac{2}{3}$ of the minute divisions, and it is between nine and ten o'clock; find the exact time indicated by the watch.

3. Two dice are placed together so as to form a parallelopiped; determine the chance that two or more contiguous faces of the dice will have the same marks.

4. If $P_r = (a_1 - a_r)(a_2 - a_r)\ldots(a_{r-1} - a_r)(a_{r+1} - a_r)\ldots(a_n - a_r)$, and $S = \frac{a_1^m}{P_1} + \frac{a_2^m}{P_2} + \ldots + \frac{a_n^m}{P_n}$ then $S = \frac{1}{a_1 a_2 \ldots a_n}$ when $m = -1$; $S = 0$ when m is any whole number from 0 to $n-2$; and $S = (-1)^{n-1}$ when $m = n-1$; prove this generally, and when $n = 3$.

1841 1. Shew that the equation

$a^2b^4(x'-x)^2 + a^4b^2(y'-y)^2 + (b^2x^2 + a^2y^2 - a^2b^2)(b^2x'^2 + a^2y'^2 - a^2b^2) = 0$

is equivalent to the two $a^2b^2 - a^2yy' - b^2xx' = 0$, and $xy' - x'y = 0$.

2. If it be required to find the n quantities, $x_1, x_2, \ldots\ldots x_n$ from the n equations

$$x_1 + x_2 = a_1,$$
$$x_2 + x_3 = a_2,$$
$$\ldots\ldots\ldots$$
$$x_{n-1} + x_n = a_{n-1},$$
$$x_n + x_1 = a_n,$$

shew that the problem will be determinate if n be an odd number, but indeterminate or impossible if n be an even number.

3. If $a_1\, a_2 \ldots a_n$ be positive quantities, $\frac{n-1}{2}(a_1 + a_2 \ldots + a_n)$ is always greater than $\sqrt{a_1 a_2} + \sqrt{a_1 a_3} + \sqrt{a_2 a_3} + \ldots\ldots$Also, the former expression is the greatest value the latter can have under the condition $a_1 + a_2 + \ldots + a_n =$ constant.

4. If there be n equations between n unknown quantities $x_1\, x_2 \ldots x_n$, and $n+1$ known quantities $a_1\, a_2 \ldots a_n$ and c, the m^{th} equation being

$$a_1^{m-1}x_1 + a_2^{m-1}x_2 \ldots + a_n^{m-1}x_n = c^{m-1};\ \text{then}$$

$$x_1 = \frac{(c-a_2)(c-a_3)\ldots(c-a_n)}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)},\ \text{and similarly for } x_2, x_3 \ldots$$

Prove this when $n = 3$, or generally.

5. A telegraph has m arms, and each arm is capable of n distinct positions; find the total number of signals which can be made with the telegraph.

✓6. A hollow square and a hollow equilateral wedge, each r deep, have the same number n of men in each of their fronts; also the number of men in the innermost ranks of the square is equal to p times the corresponding number in the wedge; shew that $r(9p-8) = n(3p-4) + 6p - 4$.

7. There are four candidates for an office and four electors; each elector gives two votes, one to each of two candidates; and in the first scrutiny one candidate, viz. he who has fewest votes or none, is to be rejected. Shew that the rejected candidate may not be the one who is least approved. Also, neglecting moral causes, find the chance of its being necessary to repeat the first scrutiny, (1) on account of each of the four candidates having two votes, and (2) on account of some two at least having the same number of votes or none.

8. Finite Differences, Prob. 1.

9. Between a and b, a being $< b$, insert n means, a_1, a_2, $\ldots\ldots a_n$, such that $a_1 - a$, $a_2 - a_1$, $\ldots\ldots b - a_n$ form an arithmetical progression whose common difference is d; and find the limits between which the value of d must lie.

✓ 10. There are n straight lines in a plane; α of them are parallel to each other and to none else; β parallel to each other and to none else, and so on; and none of the remaining $n - \alpha - \beta - \gamma - \ldots$ lines are parallel to any of the former, or to each other; find the greatest possible number of their intersections.

1. Prove that, if $\frac{p}{q}, \frac{p_1}{q_1}, \frac{p_2}{q_2}$ be three consecutive convergents of a continued fraction, 1842

$$\frac{p_1}{q_1} = \frac{p_2 - p}{q_2 - q}.$$

2. If $C_r(n)$ be the number of combinations, and $P_r(n)$ the number of homogeneous products of r dimensions which can be made with n letters, shew that, so long as $n - m < r$,

$$P_r(m) - C_1(n)\, P_{r-1}(m) + C_2(n)\, P_{r-2}(m) - \&c. \pm C_r(n) = 0.$$

3. Shew that when n is any prime number greater than 2,

$x^n - 1$ when divided by n will leave the same remainder as $(x-1)^n$ divided by n.

1843 1. A certain fraction becomes 1, when 3 is added to its numerator; and becomes $\frac{1}{2}$ when 2 is added to its denominator: what fraction is it?

2. If $y = e^{\frac{1}{1-\log x}}$, and $z = e^{\frac{1}{1-\log y}}$, prove that $x = e^{\frac{1}{1-\log z}}$.

3. Shew that $e^{e^{\cdot^{\cdot^{\cdot^{e}}}}}$ (x indices.) $= \log^{-n}(e)$.

4. If any number (n) can be resolved into the sum of (q) squares, $2(q-1)n$ can be resolved into the sum of $q(q-1)$ squares.

1844 1. Calling March the first month of the year, shew that, if the m^{th} day of the $(n+1)^{\text{th}}$ month be the y^{th} day of the year, then

$$y = m + 31n - p,$$

where p is the greatest integer contained in $\frac{5n+4}{12}$.

Supposing the first of March to be Monday, find hence on what day of the week the 10th of October will fall.

2. A man is to receive a certain number of shillings, he knows that the digits of the number are 1, 2, 3, 4, 5, but he is ignorant of the order in which they stand; determine the value of his expectation.

3. The sum of all the numbers which contain the same digits is divisible by the sum of those digits.

4. If C_r denote any combination by addition of n quantities taken r and r together, and p is an integer less than n,

$$\Sigma C_1^p - \Sigma C_2^p + \Sigma C_3^p - \&c. = 0.$$

What does this result become when all the quantities are equal?

5. A person attempts to perform a feat of dexterity, his probability of success at any trial being m. Shew that, on the

long run, he will succeed $\frac{m}{1-m}$ times in succession, without a failure.

6. If $f(pq)$ is the co-efficient of x^q in $(1+x+\ldots+x^n)^p$, show by means of the equation

$$\log(1+z) = z - \tfrac{1}{2}z^2 + \&c., \text{ that}$$

$$f(1,q) - \tfrac{1}{2}f(2,q-1) + \&c. = \frac{1}{q+1}, \text{ unless } q+1 \text{ is a}$$

multiple of $n+2$.

7. If the expression

$(a_1x_1 + a_2x_2 + \ldots + a_nx_n)^2 + (b_1x_1 + b_2x_2 + \ldots + b_nx_n)^2 + \&c.$ to n terms, be reduced, by means of n linear relations connecting $x_1x_2\ldots x_n$ with $u_1u_2\ldots u_n$, to the form $A_1u_1^2 + \ldots + A_nu_n^2$, then $\{A_1A_2\ldots A_n\}$ is a perfect square.

1. Two persons A and B walk to meet each other from 1845
the extremities of a line AB; a third person C, who walks faster than either A or B, starts with A, and when he meets B turns back, and so on till they all come together; the distance C walks in the direction AB is twice that which he walks in the contrary direction, and when they all come together he has passed over a space equal to AB; shew that the rates of A, B, C are as 1, 2, 3.

2. The co-efficient of x^r in the expansion of $(1+x)(1+cx)(1+c^2x)(1+c^3x)\ldots$, the number of factors being unlimited, and c less than unity, is equal to

$$\frac{c^{\frac{1}{2}r(r-1)}}{(1-c)(1-c^2)(1-c^3)\ldots(1-c^r)}.$$

3. If S_r represents the series $1^r + 2^r + 3^r + \ldots + (p-1)^r$, and $a_1a_2\ldots$ are the co-efficients of $x\,x^2\ldots$ in the expansion of $(1+x)^{n+1}$,

$$p(p^n - 1) = a_1S_1 + a_2S_2 + \ldots + a_2S_{n-1} + a_1S_n.$$

1. Prove that 1846

$(A^2 + B^2 + C^2 + \ldots)(a^2 + b^2 + c^2 + \ldots) > (Aa + Bb + Cc + \ldots)^2$, unless A, B, $C\ldots$ are proportional to a, b, $c\ldots$

2. A given number of persons take wine together, each bowing to each of his friends in succession; if all the persons

bow simultaneously each time, find the chance that a given person shall bow to a given number of his friends just as they are bowing to him.

3. A person engages in play against a bank, playing under a slight disadvantage, but retaining the option, which is not allowed to the bank, of ceasing to play when he pleases, and he determines accordingly to cease after a certain run of luck; shew by general reasoning that no possible arrangement on his part can throw the advantage on his side, or even diminish the odds against him.

4. There are n planes of which no two are parallel to each other, no three parallel to the same right line, and no four pass through the same point; prove that the number of lines of intersection of the planes is $\frac{1}{2} n (n-1)$, and that the number of points of intersection of those lines is $\frac{1}{6} n (n-1) (n-2)$.

1847 1. Prove that

$$(ax_1^2 + 2bx_1y_1 + cy_1^2)(ax_2^2 + 2bx_2y_2 + cy_2^2) - \{ax_1x_2 + b(x_1y_2 + x_2y_1) + cy_1y_2)\}^2 = (ac - b^2)(x_1y_2 - x_2y_1)^2.$$

2. A boat's crew row $3\frac{1}{2}$ miles down a river and back again in $1^{h}.\ 40^{m}.$; supposing the river to have a current of 2 miles per hour, find the rate at which the crew would row in still water.

3. If a and x both lie between 0 and 1, prove that $\frac{1-a^x}{1-a} > x$.

4. Prove that the number of ways in which any number x can be composed of n numbers (not necessarily different from each other), is equal to the number of ways in which x can be composed of n and numbers not exceeding n, the order in which the numbers occur not being considered.

5. If all the odd numbers be arranged in order of magnitude, and then separated into groups, so that the first group contains one number, the second two, and so on, prove that the sum of the numbers in any group is equal to the cube of the natural number which marks the order of the group.

1. * A ship sails with a supply of biscuit for 60 days, at a 1848
daily allowance of 1 lb. a head; after being at sea 20 days she encounters a storm, in which 5 men are washed overboard, and damage sustained that will cause a delay of 24 days, and it is found that each man's allowance must be reduced to $\frac{5}{7}$ lb.; find the original number of the crew.

2. * If a, b, and x be positive, and $a > b$, prove that
$\frac{x+a}{\sqrt{(x^2+a^2)}} > < \frac{x+b}{\sqrt{(x^2+b^2)}}$ according as $x > < \sqrt{ab}$.

3. * If a, b, c be in harmonic progression, and n be a positive integer, shew that $a^n + c^n > 2b^n$.

4. A walks to Trumpington and back by Grantchester in $1\frac{1}{4}$ hr., starting between 2 o'clock and $2\frac{3}{4}$, B walks the same distance in the same direction in $1\frac{1}{2}$ hr., starting between 2 and $2\frac{1}{4}$; find the chance that A overtakes B before he gets home.

5. A parallelopiped is cut by three systems of parallel planes, given in number, parallel to the three pair of opposite faces respectively; find the total number of parallelopipeds formed in everyway.

6. If $(\beta - \alpha)(y - m\alpha) = (n\beta - m\alpha)(x - \alpha)$,
and $(\beta' - \alpha')(y - m\beta') = (m\beta' - n\alpha')(x - \beta')$,
shew that $\left(\frac{1}{\alpha\alpha'} - \frac{1}{\beta\beta'}\right) x = \frac{\alpha + \alpha'}{\alpha\alpha} - \frac{\beta + \beta'}{\beta\beta'}$.

7. Shew that the integral parts of $(\sqrt{3}+1)^{2m+1}$ and $(\sqrt{3}+1)^{2m}+1$ are respectively divisible by 2^{m+1} where m is any integer whatever.

8. Prove that for a given integral value of a there are

(1) a integral values of b which will make the integral part of $(a+\sqrt{b})^{2m+1}$ divisible by 2^{m+1}:

(2) a integral values of b which will make the integral part of $(a+\sqrt{b})^{2m+1}+1$ divisible by 2^{m+1}:

(3) $2a$ integral values of b which will make the integral part of $(a+\sqrt{b})^{2m}+1$ divisible by 2^{m+1}.

1849 1. * Reduce to its simplest form the expression

$$\frac{(1-a^2)(1-b^2)(1-c^2)-(c+ab)(b+ca)(a+bc)}{1-a^2-b^2-c^2-2abc}.$$

2. * Find a whole number which is greater than three times the integral part of its square root by unity. Shew that there are two solutions of the problem and no more.

3. A quantity of corn is to be divided amongst n persons, and is calculated to last a certain time if each of them receive a peck every week: during the distribution it is found that one person dies every week and then the corn lasts twice as long as was expected; find the quantity of corn and the time that it lasts.

4. If $(m\,Cr)$ be the number of combinations of m things taken r together, and p be less than m and n, shew that

$$\Sigma_{r=0}^{r=p}\{(m\,Cr)(n\,C\overline{p-r})\} = (\overline{m+n}\,Cp).$$

5. If $(n\,Cr)$ be the number of combinations of n things taken r together, and a be an integer > 1 then will

$$(n\,a\,Cr\,a) \text{ be } > (n\,Cr)^a.$$

6. If p be greater than unity,

$$\frac{1}{p-1} = \frac{1}{p+1} + \frac{1.2}{(p+1)(p+2)} + \frac{1.2.3}{(p+1)(p+2)(p+3)} + \ldots\ldots$$

and if it be less,

$$\frac{1}{1-p} = \frac{1}{1+p} + \frac{1.2}{(1+p)(1+2p)}p + \frac{1.2.3}{(1+p)(1+2p)(1+3p)}p^2 + \ldots.$$

7. A bag contains three bank notes, and it is known that each of them is either a £5, £10, or £20 note; at three successive dips into the bag (replacing the note after each dip) a £5 note was drawn: what is the probable value of the contents of the bag?

THEORY OF EQUATIONS.

1839 If we know two numbers which *nearly* satisfy two equations $\phi(x, y) = 0, f(x, y) = 0$; shew how to approximate to the true roots of those equations by a method analogous to that of Newton.

If k be the last term of an equation of the n^{th} degree whose roots are in geometric progression, shew that $k^{\frac{1}{n}}$ is one of the roots, n being odd. Shew that in a similar manner one root of an equation of odd degree, whose roots are either in Arithmetic or Harmonic Progression may be found. 1843

Prove that the equation $(x-1)\,\epsilon^{x} = (x+1)\,\epsilon^{-x}$ has two, and only two real roots, which are equal in magnitude and of opposite signs. 1846

If $x_1\ x_2 \ldots\ldots x_n$ be the roots of an algebraical equation 1849

$$f(x) = x^n + p_1 x^{n-1} + \ldots\ldots + p_n = 0,$$

and no two of them be equal, then

$$\frac{1}{p_n} + \frac{1}{x_1 f'(x_1)} + \frac{1}{x_2 f'(x_2)} + \ldots\ldots + \frac{1}{x_n f'(x_n)} = 0;$$

$$\text{and } \frac{x_1^{m-1}}{f'(x_1)} + \frac{x_2^{m-1}}{f'(x_2)} + \ldots\ldots + \frac{x_n^{m-1}}{f'(x_n)} = 0$$

m being the positive integer less than n.

PLANE TRIGONOMETRY.

1. If $a = b + c$ shew that 1838

$$\left\{\frac{1-\cos ma}{ab} + \frac{1-\cos mc}{bc}\right\}^2 = \left\{\frac{\sin ma}{ab} - \frac{\sin mc}{bc}\right\}^2$$

$$= \frac{1}{abc}\left\{\frac{a^2+b^2+c^2}{abc} + 2\,\frac{\cos ma}{a} + 2\,\frac{\cos mb}{b} - 2\,\frac{\cos mc}{c}\right\};$$

and find the value of either side of the above equation when $a = c$.

2. Write down the first 20 positive values of θ that satisfy the equation $\frac{\sin 2\theta}{\theta} \cdot \frac{\sin 12\theta}{\sin 3\theta} = 0$, and express generally the values of θ which satisfy it.

1. Prove the following equations, 1839

Sin $(\alpha+\beta)\cos\beta - \sin(\alpha+\gamma)\cos\gamma = (\beta-\gamma)\cos(\alpha+\beta+\gamma)$.

Sin $(\alpha+\beta-2\gamma)\cos\beta - \sin(\alpha+\gamma-2\beta)\cos\gamma =$

Sin $(\beta-\gamma)\ \{\cos(\beta+\gamma-\alpha) + \cos(\alpha+\gamma-\beta) + \cos(\alpha+\beta-\gamma)\}$.

2. If, in a plane triangle, $b - a = m \cdot c$; prove that

$$A = \cos^{-1}\left(m \cdot \cos\frac{C}{2}\right) - \frac{C}{2}, \text{ and } \cot\frac{B-A}{2} = \frac{1 + m \cdot \cos B}{m \cdot \sin B}.$$

3. If (r) be the radius of the circle inscribed between the *base* of a right-angled triangle and the other two sides produced, and (r') be the radius of the circle inscribed between the *altitude* of the same triangle and the other two sides produced; the area of the triangle will be equal to the rectangle $r \cdot r'$.

4. If (r) and (r') be the radii of two circles in one plane and (a) the distance between their centers, and the line joining those centers be produced indefinitely; the two points in this line, from each of which the two circles will *appear* equally large, are at a distance $\frac{2arr'}{r'^2 - r^2}$ from each other; and, if a circle be described upon the line joining these two points as a diameter, the two original circles will appear equally large from every point in the circumference of this third circle.

1840 $\surd$ 1. Eliminate ϕ and i from the equations

$$A = \left(\frac{\sin^2\phi}{a^2} + \frac{\cos^2\phi}{b^2}\right)\cos^2 i + \frac{\sin^2 i}{c^2}, \; B = \frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2},$$

$$\text{and } C = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\sin\phi\cos\phi\cos i.$$

2. Shew that the co-efficient of x^p in the expansion of

$$\overline{1 - 2\cos\phi \cdot x + x^2}\,]^{-n} = 2\left\{a_p \cos p\phi + a_1 a_{p-1}\cos(p-2)\phi + a_2 a_{p-2}\cos(p+4)\phi + \&c.\right\},$$

where a_m generally equals the co-efficient of x^m in the expansion of $(1 - x)^{-n}$.

3. Shew that

$$(\cos\theta)^m + \left\{\cos\left(\frac{2\pi}{n} + \theta\right)\right\}^m + \left\{\cos\left(\frac{4\pi}{n} + \theta\right)\right\}^m + \ldots + \left\{\cos\left(\frac{2(n-1)\pi}{n} + \theta\right)\right\}^m$$

is independent of θ when m is less than n; but if r be the

greatest integer contained in $\frac{m}{n}$, the above series can be reduced to the form

$$A_0 + A_1 \cos n\theta + A_2 \cos 2n\theta + \ldots + A_r \cos rn\theta$$

where $A_0, A_1 \ldots A_r$ are independent of θ.

Exs. $n = 3, m = 2$; $n = 3, m = 5$.

4. Three indefinite straight lines intersect in A, B, and C; any other straight line cuts AB in C', BC in A', and CA in B'; Then $AB' \cdot BC' \cdot CA' = A'B \cdot B'C \cdot C'A$, and the product of the areas of the triangles

$$A'BC', B'CA', C'AB' = \frac{(A'B' \cdot B'C' \cdot C'A' \sin A' \sin B' \sin C')^2}{8 \sin A \sin B \sin C}.$$

5. Assuming the known theorem

$$a = b \cos (ab) + c \cos (ac) + d \cos (ad) + \&c.,$$

where a, b, c, &c. are the sides of a plane polygon, (ab) the angle between a and b, &c., prove that the sum of the squares of all the sides, is equal to twice the sum of the products of every two sides and the cosine of the angle between them: i. e. $\Sigma a^2 = 2\Sigma ab \cos (ab)$.

6. A complete pyramidal pile of equal shot upon a square horizontal base, has four shot in each side of the lowest tier. Find the dimensions of the pyramid that envelopes and contains the whole pile.

7. If the instrument be perfect by which, in surveying, a series of points are laid down, from the observed angles which the distances between three stations in the same plane, whose positions are accurately known, subtend at each point; shew how an error in one of the observed angles affects the position of the point to be laid down. Prove that the displacement cannot be very great in comparison with the error of observation, unless the difference between the distances of two of the fixed stations be very small. Point out how, in a given position, to estimate the displacement caused by an error in the position of one of the fixed stations.

1841 1. Having given $\tan\phi = \dfrac{\sin\theta\cos\theta'}{\sin\theta' + \cos\theta}$, shew that

$$\tan\frac{\phi}{2} = \tan\frac{\theta}{2}\tan\left(45^0 - \frac{\theta'}{2}\right).$$

2. What plane rectilineal figure is that in which a similar rectilineal figure can be inscribed, whose area shall be $\left(\frac{1}{n}\right)^{\text{th}}$ of the area of the former?

Ex. $n = 2$, $n = 4$.

3. A number of spheres are put into a hollow cone one above another, and each touching the one above, the one below, and the sides of the cone. The radii of the spheres form a geometrical progression of which the common ratio is $\tan^2$ ($45^0 + \frac{1}{4}$ the vertical angle of the cone).

4. If from one of the angles of a rectangle a perpendicular be drawn to its diagonal, and from the point of their intersection lines be drawn perpendicular to the sides which contain the opposite angle; shew that if p and p' be the lengths of the perpendiculars last drawn, and d the diagonal of the rectangle, $p^{\frac{2}{3}} + p'^{\frac{2}{3}} = d^{\frac{2}{3}}$.

5. Shew how to find the distance of an inaccessible object upon a horizontal plane by means of a measuring chain, or staff alone, and without observing any angles.

6. If $\sin(a + ax) = c\tan(b + \beta x)$ for all values of x; then, so long as ax and βx are small quantities, x is very nearly

$$= \frac{a\cot a - 2\beta\operatorname{cosec} 2b}{a^2 + 2\beta^2\sec^2 b}.$$

1842 1. If $\cos^2\theta = \dfrac{\cos a}{\cos\beta}$, $\cos^2\theta' = \dfrac{\cos a'}{\cos\beta}$, and $\dfrac{\tan\theta}{\tan\theta'} = \dfrac{\tan a}{\tan a'}$;

shew that $\tan\dfrac{a}{2}\tan\dfrac{a'}{2} = \tan\dfrac{\beta}{2}$.

2. If the sines of the angles of a triangle be in arithmetical progression, the co-tangents of the half-angles are also in arithmetical progression.

3. If a, b, c be the sides, A, B, C the angles of a triangle,

$$a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C:$$

R being the radius of the circumscribing circle.

4. If $p_1, p_2, \ldots p_{2n-1}, p_{2n}$ be the perpendiculars drawn from any point in the circumference of a circle on the sides of a regular circumscribing polygon of $2n$ sides, shew that

$$p_1 \cdot p_3 \cdots p_{2n-1} + p_2 \cdot p_4 \cdots p_{2n} = \frac{a^n}{2^{n-2}}.$$

where a is the radius of the circle.

1. If two circles, whose radii are a and b, touch each other externally, and if θ be the angle contained by the two common tangents to these circles; shew that $\sin\theta = \frac{4(a-b)\sqrt{ab}}{(a+b)^2}$. 1843

2. $$\tan\theta + \tan\left(\frac{2\pi}{m} + \theta\right) + \ldots\ldots + \tan\left(2\frac{m-1}{m}\pi + \theta\right) = \frac{m \sin m\theta}{\cos m\theta - S_m (-1)^{\frac{m}{2}}},$$

where $S_m = \frac{1}{2}$. the sum of the m^{th} powers of the roots of $z^2 - 1 = 0$.

1. Eliminate θ from the equations 1844

$$(a+b)\tan(\theta-\phi) = (a-b)\tan(\theta+\phi),\quad a\cos 2\phi + b\cos 2\theta = c.$$

2. The sides of a certain triangle are proportional to the roots of the equation $x^3 + p_1x^2 + p_2x + p_3 = 0$; find the sum of the cosines of its angles.

1. Find in a form adapted to logarithmic computation the distance of two points CD in the same plane with a given base AB, the angles subtended by AB, AD at C, and by AB, BC at D being given. 1845

2. Straight lines drawn through the angular points of a triangle divide the sides in the ratios a to d, b to e, and c to f. Shew that the ratio of the area included by these lines to the area of the triangle is equal to

$$\frac{(abc - def)^2}{(ab + ae + de)(ac + cd + df)(bc + bf + ef)}.$$

3. If two circles can be described so that each shall touch the other and three sides of a quadrilateral figure, one-fourth of the difference of the sums of the opposite sides is a mean proportional between the radii.

Express the area of the quadrilateral figure in terms of the sides and the radii of the circles.

1846 1. From the identical equation

$$\frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(c-b)} = 1$$

deduce the following, by assuming $x = \cos 2\theta + \sqrt{-1} \sin 2\theta$, and corresponding expressions for a, b and c;

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)} \sin 2(\theta-\alpha) + \frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)} \times$$

$$\sin 2(\theta-\beta) + \frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)} \sin 2(\theta-\gamma) = 0.$$

2. Three circles touch each other externally; prove that the square of the area of the triangle formed by joining their centres is equal to the product of the sum and product of their radii.

3. The sky being covered by a level stratum of clouds at the height of a mile above the Earth's surface, shew that the distance in miles of the edge of a cloud whose altitude is a, is very nearly $\frac{1}{\sin a}\left(1 - \frac{\cot^2 a}{2n}\right)$, unless the altitude be very low, n being the number of miles in the Earth's radius.

4. The angles of two circular sectors of equal area and having a common chord (c) are as 3 : 1; shew that the lune which the arcs of the sectors include $= \frac{c^2}{6}\sqrt{3+2\sqrt{3}}$.

1847 1. If r be the radius of the circle inscribed in a triangle, and r_a, r_b, r_c the radii of the circles inscribed between this circle and the sides containing the angles A, B, C respectively prove that

$$\sqrt{r_b r_c} + \sqrt{r_c r_a} + \sqrt{r_a r_b} = r.$$

2. When x is any prime number, prove that the integral part of $\left(\cot\frac{\pi}{8}\right)^x - 2$ is divisible by $4x$, and that of $\left(\cot\frac{\pi}{12}\right)^x - 2^{x+1} + 1$ is divisible by $6x$.

Also, when x and y are any odd numbers, prove that the integral part of $(\sqrt{3})^y\left(\cot\frac{\pi}{12}\right)^x$ is divisible by 6 as long as y is less than x; and if x be given find the least value of y which makes the integral part of $(\sqrt{3})^y\left(\cot\frac{\pi}{12}\right)^x$ not divisible by 6.

1. * The angles of a quadrilateral inscribed in a circle, 1848
taken in order, when multiplied by 1, 2, 2, 3 respectively, are in arithmetical progression, find their values.

2. * Prove that $\sin 3\theta \sin^3\theta + \cos 3\theta \cos^3\theta = \cos^3 2\theta$.

3. A cannon ball is moving in a direction making an acute angle θ with a line drawn from the ball to an observer; if V be the velocity of sound, and nV that of the ball, prove that the whizzing of the ball at the different points of its course will be heard in the order in which it is produced, or in the reverse order, according as $n < > \sec.\ \theta$.

4. If $\frac{\sin(\alpha-\beta)}{\sin\beta} = \frac{\sin(\alpha+\theta)}{\sin\theta}$ shew that

$$\cot\beta - \cot\theta = \cot(\alpha+\theta) + \cot(\alpha-\beta).$$

1. * If $\phi - \alpha$, ϕ, $\phi + \alpha$ be three angles whose cosines are in 1849
harmonical progression, shew that

$$\cos\phi = \sqrt{2}.\cos\frac{\alpha}{2}.$$

2. * A person, wishing to ascertain his distance from an inaccessible object, finds three points in the horizontal plane at which the angular elevation of the summit of the object is the same. Show how the distance may be found.

3. If $\cos\alpha = \cos\beta\ \cos\phi = \cos\beta'\cos\phi'$,

$$\text{and } \sin\alpha = 2\sin\frac{\phi}{2}.\sin\frac{\phi'}{2}.$$

shew that $\tan \frac{a}{2} = \tan \frac{\beta}{2} \cdot \tan \frac{\beta'}{2}$.

4. Find $\frac{a}{b}$ from the equations,

$$(a+b)\sin\theta + (a-b)\cos\theta = \sqrt{a^2+b^2},$$
$$a\sin^3\theta + b\cos^3\theta = \sqrt{3ab}.$$

SPHERICAL TRIGONOMETRY.

1838 1. If O be the middle point of the equilateral spherical triangle XYZ, and P any point upon the surface of the sphere such that

$$h \sec PX = k \sec PY = l \sec PZ,$$
$$(\tan PO \tan XO)^2 = 4\,\frac{h^2+k^2+l^2-kl-lh-hk}{(h+k+l)^2}.$$

2. Shew that if O be any point within a spherical triangle ABC, the product of the sines of any two sides and the sine of the included angle
$= \sin AO \sin BO \sin CO\,[\cot AO \sin BOC + \cot BO \sin COA + \cot CO \sin AOB]$.

3. The base of a pyramid is an equilateral six-sided figure, the angles of which are alternately greater and less, any two alternate angles being equal. Having given the angles between any face of the pyramid and each of two adjacent faces; find the angle between a normal to any face and the axis of the pyramid, in a form adapted to logarithmic computation.

1840 1. If a, β, γ, be the angles which three diameters of a sphere make with each other, $\sigma = \frac{a+\beta+\gamma}{2}$, and a be the radius of the sphere, then the volume of the parallelopiped formed by planes which touch the sphere at the extremities of the three diameters

$$= \frac{4a^3}{\sqrt{\sin\sigma \sin(\sigma-a)\sin(\sigma-\beta)\sin(\sigma-\gamma)}}.$$

2. If r be the radius of the small circle inscribed in a

spherical triangle, R the radius of the circumscribed circle, r_1, r_2, r_3 the radii of the three circles each of which touches one side of the triangle and the other two sides produced, then $\cot r_1 + \cot r_2 + \cot r_3 - \cot r = 2 \tan R$. Prove this, and deduce the corresponding expression for a plane triangle.

3. The axes of any three possible zones of a crystal which are not in one plane, may be employed as crystallographic axes.

If a point on a sphere be referred to two great circles, at 1842
right angles to each other, as axes, by means of the portions of these axes cut off by great circles drawn through the point and two points on the axes each 90° from their intersection: shew that the equation to a great circle is

$$\frac{\tan \theta}{\tan \alpha} + \frac{\tan \phi}{\tan \beta} = 1 ;$$

θ and ϕ being the co-ordinates of any point in the circle, and α and β their values when the circle cuts the axes.

1. If r_1, r_2, r_3 be the radii of three small circles of a 1843
sphere (radius r) which touch one another in P, Q, R, and A, B, C be the angles of the spherical triangle formed by joining their centres,

$$\text{area } (PQR) = (A \cos r_1 + B \cos r_2 + C \cos r_3 - \pi)\, r^2.$$

2. The stereographic projection of a cube on a plane perpendicular to its diagonal (the pole being in the diagonal produced) is an equilateral hexagon, whose angles are alternately greater and less. If the pole of projection be at a distance from the cube, equal to its diagonal, the sines of two adjacent angles of the hexagon are as 8 : 5.

Four points A, B, C and D on the surface of a sphere are 1845
joined by arcs of great circles, and E, F are the middle points of the arcs AC, BD; shew that

$\cos AB + \cos BC + \cos CD + \cos AD = 4 \cos AE \cos BE \cos FE$, and deduce the property of a plane quadrilateral stated in Euclid, Prob. I.

1846 ABC is a great circle of the sphere, AA', BB', CC' are arcs of great circles drawn perpendicular to ABC, and reckoned positive when they lie on the same side of it; prove that the condition of A', B', C' lying in a great circle is

$$\tan AA' \sin BC + \tan BB' \sin CA + \tan CC' \sin AB = 0.$$

1847 1. If through any point P, within a spherical triangle ABC, great circles be drawn from the angular points A, B, C, to meet the opposite sides in a, b, c respectively, prove that

$$\frac{\sin Pa \cdot \cos PA}{\sin Aa} + \frac{\sin Pb \cdot \cos PB}{\sin Bb} + \frac{\sin Pc \cdot \cos PC}{\sin Cc} = 1,$$

and thence deduce the corresponding property of a plane triangle.

2. ABC, $A'B'C'$ are two equal spherical triangles, the equal angles being placed in the same order; if the vertices of the equal angles be joined by arcs of great circles AA', BB', CC', prove that the great circles which bisect these arcs at right angles meet in a point, and that AA', BB', CC' subtend equal angles at that point.

3. OAA_1, is a spherical triangle, right-angled at A_1; the arc $A_1 A_2$ of a great circle is drawn perpendicular to OA, $A_2 A_3$ is drawn perpendicular to OA_1, and so on; prove that $A_n A_{n+1}$ vanishes when n becomes infinite, and, find the value of $\cos AA_1 \cdot \cos A_1 A_2 \cdot \cos A_2 A_3 \ldots$ *to infinity*.

1848 1. In a right-angled spherical triangle, shew that

$$\sin a \tan \frac{A}{2} \sin b \tan \frac{B}{2} = \frac{2 \sin^2 \frac{1}{2}(a+b+c)}{\sin (a+b)}.$$

Shew also that if E be the spherical excess

$$\sin \frac{E}{2} = \frac{\sin \frac{a}{2} \sin \frac{b}{2}}{\cos \frac{c}{2}}; \qquad \cos \frac{E}{2} = \frac{\cos \frac{a}{2} \cos \frac{b}{2}}{\cos \frac{c}{2}}.$$

2. If three small circles be inscribed in a spherical triangle, having each of its angles 120°, so that each touches the other two as well as two sides of the triangle, prove that the radius of

each of the small circles = 30°, and that the centres of the three circles coincide with the angular points of the polar triangle.

If P be the perimeter of a spherical triangle, of which 1849
the angles are A, B, C, and the spherical excess E, prove that

$$\sin\frac{P}{2} = \frac{\left\{\sin\frac{E}{2}\sin\left(A-\frac{E}{2}\right)\sin\left(B-\frac{E}{2}\right)\sin\left(-C\,\frac{E}{2}\right)\right\}^{\frac{1}{2}}}{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}.$$

GEOMETRY OF TWO DIMENSIONS.

1. HKL, PQR are two triangles, prove that if the straight 1838
lines HP, KQ, LR meet in one point, the intersection of KL, QR; LH, RP; HK, PQ lie in a straight line.

2. If on any three chords drawn through the same point in the circumference of a circle, as diameters, three circles be described: the points of intersection of these circles two and two lie in the same straight line. Prove this, and find the equation to that line.

3. If $y = mx$ be a chord of a circle whose radius is (r): shew that the equation to the circle whose diameter is $y = mx$, is

$$(1+m^2)(x^2+y^2) - 2r(x+my) = 0.$$

4. If CP be any semi-diameter of an ellipse, and AQO be drawn from the extremity of the major axis parallel to CP, and meeting the curve in Q and the minor axis produced in O: shew that

$$2CP^2 = AO \cdot AQ.$$

5. A quadrant of an ellipse is inscribed in a right-angled triangle having its axes coincident with the sides containing the right-angle; find the curve traced out by its center of curvature at the point of contact with the hypothenuse, while the point of contact moves from one acute angle to the other.

1. If a rod move betwixt two fixed straight lines CP, CQ; 1839
and the perpendiculars from P, Q upon CP, CQ respectively

meet in R, whilst those from P, Q upon CQ, CP respectively meet in S; the loci of R and S will be two circles having their common centre in C.

2. The locus of the vertices of all similar triangles whose bases are parallel chords of a parabola, will in general be another parabola: but if any one of the triangles touch the parabola with both its sides, the locus becomes a straight line.

3. If two parabolas be described with a common focus (S), passing through a common point (A), and having the sum of the inclinations of their axes to the line SA equal to two right-angles; their second point of intersection will be somewhere in the straight line drawn through (A) at right-angles to SA.

4. If any number of circles be described so as to touch a given curve of the second order in one given point, they will each of them generally cut it in two others; and if chords be drawn between the two points of section for each circle, these chords will all be parallel to one another.

Prove this, and from the investigation deduce the radius of curvature at the given point.

5. Two relations being given between x, y and a subsidiary quantity ϕ; the elimination of ϕ will give a relation between x and y which is not necessarily equivalent to the original equations. Illustrate this by tracing the curves determined by the subsidiary angles in the following examples:—

(1) $x = a\left\{\dfrac{1-(\cos\phi)^4}{1+(\cos\phi)^4}\right\}$, $y = \dfrac{2b(\cos\phi)^2}{1+(\cos\phi)^4}$; where $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 = 1$.

(2) $x = a(\cos\phi)^4$, $y = b(\sin\phi)^4$: where $\sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{y}{b}} = 1$.

(3) $x = a\left\{\cos\phi + \sqrt{(\cos\phi)^2 - n^2}\right\}$,

$y = b\left\{\cos\phi - \sqrt{(\cos\phi)^2 - n^2}\right\}$; where $xy = abn^2$.

1840 1. If a point (C') be taken in any one (as AB) of three indefinite straight lines that intersect in A, B, and C; and lines (as $C'B'A'$) be drawn from C' cutting AC, BC (as in B', A');

then all the intersections of each pair of lines (as BB', AA') drawn from B and A to the points of section (B', A'), lie in a line that passes through C.

2. Two straight lines which coincide in their initial position, revolve uniformly with different angular velocities about two fixed points; find the locus of their points of intersection, and trace the curve when the angular velocity of one line is twice that of the other.

If these lines be one pair only of several, arranged like the spokes of a wheel about the fixed points, find, for any given position of the first pair, the curve passing through the simultaneous intersections of all the pairs, formed by taking lines that have any equal angular distances from the first.

3. If PT, QT be two tangents at the points P and Q of a parabola whose focus is S, then $SP \cdot SQ = ST^2$; and if SP, SQ include a given angle α, the locus of T will be a hyperbola whose eccentricity $= \sec \frac{\alpha}{2}$.

4. If θ be the angle which the focal distance to any point of an ellipse makes with the tangent, and ϕ the angle between the lines drawn from that point to the extremities of the axis major, then

$$2 \tan \theta = e \tan \phi.$$

5. There will always be an ellipse and hyperbola, whose major axes are at right angles, which have a contact of the third order with a curve at any proposed point, and whose eccentricities e, e' are connected by the equation

$$\frac{1}{e^2} - \frac{1}{e'^2} = 1.$$

6. The foci of all those elliptical sections of a right cone, whose vertical angle is α, which have the same eccentricity e, will lie in two conical surfaces, the tangent of the sum of whose semivertical angles $= 2e \tan \frac{\alpha}{2}$. Prove this and find the vertical angle of each cone.

7. When an oblique cone upon an elliptical base (whose semiaxes are α, β and center C) admits of a circular section, if D be the foot of the perpendicular drawn from the vertex upon the base, and $CD = a$, the centers of all the circles will lie in two straight lines which meet the base in two points the sum of the reciprocals of whose distances from $C = \frac{2a}{a^2 - \beta^2}$.

8. The tangents to the interior of two concentric and similar curves of the second order whose axes are coincident, cut off from the exterior curve equal areas.

1841 1. A straight line of given length $2c$ is made to move so that its ends are always in contact with two other straight lines which include a given angle $2a$; shew that the locus of its middle point is an ellipse whose semiaxes are $c \tan a$ and $c \cot a$, and the direction of one of its axes bisects the angle included by the given straight lines.

2. If θ be the acute angle between the tangent and focal distance at any point of an ellipse, the distance of that point from the centre is

$$= \sqrt{a^2 - b^2 \cot^2 \theta}.$$

3. If $\alpha\beta$, $\alpha'\beta'$ be the co-ordinates of two points in a diameter of an ellipse, and be subject to the condition $a^2b^2 = a^2\beta\beta' + b^2\alpha\alpha'$, where a and b are the semi-axes of the ellipse, shew that the equations to the tangents at the extremities of this diameter are $a^2 \sqrt{\beta\beta'}\, y + b^2 \sqrt{\alpha\alpha'}\, x = \pm a^2b^2$.

4. If A and B be the extremities of the axis major of a conic section, T the point where a tangent at a point P in the curve meets AB, QTR a line perpendicular to AB and meeting AP and BP in Q and R respectively; then $QT = RT$.

5. If the condition that an equation of the second order may represent a parabola be introduced into the equation

$$(a^2\beta^2 + b^2\alpha^2 - a^2b^2)\,(a^2\beta_1^2 + b^2\alpha_1^2 - a^2b^2)\,(a^2y^2 + b^2x^2 - a^2b^2) + a^4b^4\,\{y\,(\alpha_1 - \alpha) - x\,(\beta_1 - \beta) + \alpha\beta_1 - \alpha_1\beta\}^2 = 0,$$

shew that the equation will be reduced to the form

$$\frac{a}{b}\sqrt{\beta\beta_1}y + \frac{b}{a}\sqrt{aa_1}x = \pm ab.$$

1. Shew that the length of the longer normal, drawn from 1842
a point in the minor axis of an ellipse at a distance c from its centre, and intercepted between that point and the curve = $\sqrt{a^2 + \frac{c^2}{e^2}}$.

2. A fixed circle is cut by a series of circles, all of which pass through two given points: shew that the lines which join the points of intersection of the fixed circle with each circle of the series all converge to one point.

3. If lines drawn through any point in an ellipse to the extremities of any diameter meet its conjugate (CD) in points M, N, prove that

$$CM \cdot CN = CD^2.$$

4. From the extremities of any chord PQ of a conic section, draw the tangents PT, QT to meet in T; then if α be the angle which the chord PQ makes with its corresponding diameter, $\cot TPQ - \cot TQP = 2 \cot \alpha$.

5. If a line be drawn from the focus of an ellipse whose eccentricity = e, so as to make a given angle α with the tangent; shew that the locus of its intersection with the tangent will be a circle which touches or falls entirely without the ellipse according as $\alpha >$ or $< \cos^{-1} e$.

6. If a right cone be cut by a plane in a hyperbola, the angle (ϕ) between the asymptotes of the section is determined by the equation

$$\cos\frac{\phi}{2} = \frac{\cos\alpha}{\sin\theta},$$

where α is the semi-vertical angle of the cone and θ the inclination of the cutting plane to the base of the cone.

7. A ball of given weight and radius is hung by a string of given length from a fixed point, to which also is attached another given weight by a string so long that the weight hangs

below the ball: find the angle which the string, to which the ball is attached, makes with the vertical.

1843 1. If two parallel lines be drawn at an inclination θ to the axis of x through two points whose co-ordinates are $a\,b$ and $a'\,b'$; shew that the distance between these lines is $(b'-b)\cos\theta-(a'-a)\sin\theta$. Hence determine the rectangle whose sides pass through four given points, and whose area is given.

2. A straight line is drawn from a point whose co-ordinates are $a\,b$, so as to touch the circle $(x^2+y^2=r^2)$, and to cut off a portion (c) from the axis of y; obtain an equation for determining c in terms of $a\,b$ and r.

Find the locus of the intersection of two tangents to a circle, such, that the rectangle under the portions (measured from the centre) cut off by them from a given diameter is always constant.

3. Two tangents $(a,\ b)$ to a parabola intersect in P at an angle (ω), and a circle is inscribed between the tangents and the curve, shew that the distance of its centre from P is

$$= \frac{a\,b}{(a+b)\sec\frac{\omega}{2}+2\sqrt{(a\,b)}\tan\frac{\omega}{2}}.$$

4. One extremity of a straight rod moves in a circle, and the other in a straight line passing through its centre; determine the equation to the path of any point of the rod, and discuss it. Determine the same when the line does not pass through the centre.

5. Shew that the equation to the locus of the middle points of all chords of the same length $(2c)$ in an ellipse is

$$c^2\cdot\frac{a^2y^2+b^2x^2}{a^4y^2+b^4x^2}+\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0.$$

1844 1. The centres of three circles (A, B, and C) are in the same right line, B and C touch A internally, and each other externally. Shew that the portion of the area of A, which is outside B and C, is equal to the area of the semicircle, described on the chord of A which touches B and C at their point of contact.

2. Three circles (A, B, and C) being circumstanced as in the preceding problem, and P, Q being the points where A is touched by B, C respectively; to find a point (R) on A, such, that the portion of the line PR, intercepted between B and A, may be equal to the portion of QR between C and A.

3. In an ellipse shew that $(SP - AC)^2 + (SD - AC)^2 = SC^2$, P and D being the extremities of any two conjugate diameters.

4. AB and BC are two right lines perpendicular to each other, A is a fixed point, B moves along a given right line, and $AB : BC$ is a given ratio: determine the locus of C.

5. If a quadrilateral be inscribable in two ellipses whose major axes are parallel or perpendicular to one another, any two of its opposite angles are together equal to two right angles.

6. If a circle, described upon a chord of parabola as diameter, meet the directrix, prove that it also touches it; and shew that all the chords for which this is possible intersect in the same point.

7. A line is drawn from the focus to any point of a conic section, and a circle is described upon it as diameter: shew that the locus of the ultimate intersections of all such circles is a circle, except, in a certain case, where it is a right line.

8. If the equation of an algebraical curve (cleared of radicals) be put in the form

$$0 = P_m + P_{m+1} + P_{m+2} \ldots\ldots + P_n \ldots\ldots (1),$$

where P_m denotes all the terms of m (the lowest) dimensions in x and y, P_{m+1} those of $m + 1$ dimensions, and so on; shew that the equation, $P_m = 0$, represents all the right lines which touch the curve at the origin. Exemplify this in the case of the equation $x^4 - 3ax^2y + 3ax^2 + y^4 = 0$.

9. In the preceding problem, if $px - qy$ be a factor of P_m which divides it only once, shew that the portion of the curve which touches the tangent ($px - qy = 0$) forms a point of contrary flexure. Also discuss the case where $px - qy$ divides P_m

twice and P_{m+1} once. (In both these cases it is assumed that $px - qy$ does not divide every term of the equation.)

10. If the equations of two curves be

$$f(xy) = 0 \ldots (1), \quad \phi(x'y') = 0 \ldots (2);$$

and if $r = \psi(\theta)$ be a result obtained by eliminating x, y, x' and y' between the equations (1), (2) and the following

$$(x'-x)^2 + (y'-y)^2 = r^2, \ (x'-x)\,dx' + (y'-y)\,dy' = 0,$$
$$dr^2 + r^2 d\theta^2 = dx^2 + dy^2:$$

shew that $r = \psi(\theta)$ is the polar equation of a curve of such a nature, that, if it be made to roll on (1), the pole will trace out (2).

1845 1. Straight lines drawn at right angles to the tangents of a parabola at the points where they meet a given straight line perpendicular to the axis, are in general tangents of a confocal parabola.

2. A triangle is inscribed in a conic section, prove that the points in which the sides of the triangle produced meet the tangents at the opposite angles are in the same straight line.

3. In a parabola whose equation is $y^2 = lx$, the ordinates of three points, such that the normals pass through the same point, are y_1, y_2, y_3; prove that $y_1 + y_2 + y_3 = 0$, and that a circle described through these points passes through the vertex of the parabola.

4. A chord (PSP') is drawn from the focus (S) of an ellipse, and the points (P, P') are joined with the other focus (H); determine the condition of the area $(PHP)'$ being a maximum. Shew that this problem is always possible.

5. The radius of the circle, which touches an hyperbola and its asymtotes, is equal to that part of the latus rectum produced which is intercepted between the curve and the asymptote.

1846 1. A, B are the centres of two equal circles, and AP, BQ are two radii which are always perpendicular to each other; find the curve which is always touched by the right line PQ, and explain the result when $AB^2 = 2AP^2$.

2. The curve whose polar equation is $r = a$ versin θ rolls on its axis, find the equation to the curve traced out by the pole; show geometrically how the latter curve comes to be a closed curve, and find its area.

3. BAB', CAC' are two circles, of radii a, b respectively, intersecting in A at an angle α; Ax, Ay are right lines bisecting the angle between the tangents at A and its supplement respectively, and the arcs AB, AC, AC' are taken equal to each other: if P, Q are the points of ultimate intersection of the lines BC, BC' with Ax, Ay respectively, as AB is indefinitely diminished, prove that the triangle PAQ is equal to

$$\frac{4 \sin \alpha \cdot a^2 b^2}{a^2 \sim b^2}.$$

4. Two lines of given lengths coincide with and move along two fixed axes, in such a manner that a circle may always be drawn through their extremities; find the locus of the centre of the circle, and shew that it is an equilateral hyperbola.

5. Prove that the normals at any two points in a parabola, on opposite sides of its axis, the ordinates of which are as 1 : 2, intersect in the evolute.

6. If two points be taken in each of two rectangular axes so as to satisfy the condition that a rectangular hyperbola may pass through all the four, shew that the position of the hyperbola is indeterminate, and that its centre describes a circle which passes through the origin and bisects all the lines which join the points two and two.

7. Fnd the locus of the centre of a circle inscribed in a sector of a given circle, one of the bounding radii of the sector remaining fixed.

8. If any chord AP through the vertex of a hyperbola be divided in Q so that $AQ : QP :: AC^2 : BC^2$, and QM be drawn to the foot of the ordinate MP, shew that a straight line at right angles to QM through the point Q cuts the transverse axis in the same ratio.

9. Two parallel tangents are drawn to a given circle A, and two circles B, C touch each other and A externally, touching also the two tangents respectively; find the locus of the point of contact of B and C.

1847 1. If the two pairs of opposite sides of a quadrilateral inscribed in a Conic Section be produced to meet, and likewise the two pairs of tangents to the curve drawn from the opposite angles of the quadrilateral, prove that the four points of intersection will be in the same straight line.

Also if from any point in the Conic Section perpendiculars be drawn to the sides of the quadrilateral, prove that the product of the perpendiculars on one pair of opposite sides is to the product of the perpendiculars on the other pair of sides in a constant ratio.

2. In any Conic Sections if PQ, PR make equal angles with a fixed chord PK, and QR be joined, prove that QR will pass through a fixed point for all positions of PQ, PR.

3. If an ellipse be projected on a plane, prove that the projections of any pair of conjugate diameters of the ellipse will be a pair of conjugate diameters of its projection.

Also, having given the ratio of the axes of the original ellipse, and the ratio of their projections on the given plane, together with the angles which those projections make with a fixed line in that plane, determine the inclination of the plane of the ellipse, to the fixed plane, and the angle which their line of intersection makes with the fixed line.

4. PT, QT are two equal tangents to a parabola, P and Q being the points of contact; if PT, QT be both cut by a third tangent, prove that their alternate segments are equal.

5. Two concentric ellipses which have their axes in the same directions intersect, and four common tangents are drawn so as to form a rhombus, and the points of intersection of the ellipses are joined so as to form a rectangle; prove that the product of the areas of the rhombus and rectangle is equal to half the continued product of the four axes.

6. A system of ellipses is described such that each ellipse touches two rectangular axes, to which its axes are parallel, and that the rectangle under the axes of the ellipses is constant; prove that each ellipse is touched by two rectangular hyperbolas, the rectangle under the transverse axes of which is equal to the rectangle under the axes of any one of the ellipses.

7. If CP, CD be any conjugate diameters of an ellipse $APBDA'$, and BP, BD be joined, and also AD, $A'P$, these latter intersecting in O, shew that $BDOP$ is a parallelogram, and that its greatest area is $ab\,(\sqrt{2}-1)$.

1. * Having given the three right lines drawn from any 1848
point to the three angular points of an equilateral triangle, determine a side of the triangle.

2. * Given the lengths of the axes of an ellipse, and the positions of one focus and of one point in the curve, give a geometrical construction for finding the centre.

3. * P is any point in an ellipse, AA' its axis major, NP an ordinate to the point P; to any point Q in the curve draw AQ, $A'Q$ meeting NP in R and S; shew that $NR \cdot NS = NP^2$.

4. * PSp is a focal chord of a parabola, RDr the directrix, meeting the axis in D; Q is any point in the curve; prove that if QP, Qp produced meet the directrix in R, r, half the latus rectum will be a mean proportional between DR, Dr.

5. In the given right lines AP . AQ are taken variable points p, q such that $Ap : pP :: Qq : qA$; prove that the locus of the point of intersection of Pq, Qp is an ellipse, which touches the given right lines in the points P, Q.

6. The corner of a sheet of paper is turned down so that the sum of the edges turned down is constant; find the equation to the curve traced out by the vertex of the angle; find also the area of the curve.

7. PSp is any focal chord of a parabola whose vertex is A; prove geometrically that AP, Ap will meet the latus rectum in two points, Q, q whose distances from the focus are equal to the ordinates of the points p and P respectively.

8. Show that the curve which trisects the arcs of all segments of a circle described upon a given base is an hyperbola whose eccentricity = 2.

9. With two conjugate diameters of an ellipse as asymptotes a pair of conjugate hyperbolas is constructed; prove that if one hyperbola touch the ellipse the other will do so likewise: prove also that the diameters drawn through the points of contact are conjugate to each other.

10. PSp is any focal chord of an ellipse, A the extremity of the axis major; AP, Ap meet the directrix in two points Q, q; show that $< QSq$ is a right angle.

11. Let D be a point in the axis minor of an ellipse whose eccentricity is e, S the focus, O the centre of curvature at the extremity of the axis minor; with centre D and radius $= \frac{DS}{e}$ describe a circle; shew that this circle will touch the ellipse or fall entirely without it according as D is nearer to or farther from the centre than the point O.

12. Tangents to a system of similar and concentric ellipses are drawn at a given perpendicular distance from the centre; find the locus of the point of contact, and shew that the area of the curve is equal to that of an ellipse which has the same greatest and least diameters.

13. From a given point in a conic section draw geometrically two chords at right angles to each other which shall be in a given ratio.

14. Determine the equation to the conic section which passes through five points whose co-ordinates are given; and thence shew that the equation to the conic section which passes through the five points whose co-ordinates are

$$1, -1;\ 2, 1;\ -2, 3;\ 3, 2;\ -1, -3 \text{ is}$$
$$61y^2 - 17xy - 65x^2 + 36y + 174x - 151 = 0.$$

15. Two chords AB, AC, are drawn from a given point A in a curve of the second order so as to contain a given angle: shew that BC will always touch a curve of the second order.

16. OA, OB, OC are three straight lines mutually at right angles, and a luminous point is placed at C: shew that when the quantity of light received upon the triangle AOB is constant, the curve which is always touched by AB will be an hyperbola whose equation referred to the axes OA, OB is $(y - mx)(x - my) = mc^2$, where $OC = c$ and m is a constant quantity.

17. P, D are the extremities of two semiconjugate diameters of an ellipse E whose semi-axes are a, b: upon PD describe an equilateral triangle PDR so that the point R may fall without the ellipse; the locus of R will be an ellipse E_1: Assuming the above result, shew that, if E_1 be similarly treated as also all the successive ellipses, the axes a_x, b_x of the x^{th} ellipse E_x, so described will be comprised in the formula

$$(a+b)\left(\cot\frac{\pi}{12}\right)^{\frac{x}{2}} \pm (a-b)\left(\tan\frac{\pi}{12}\right)^{\frac{x}{2}}.$$

1. * Draw a parabola to touch a given circle in a given point, so that its axis may touch the same circle in another given point. 1849 cc 1056.

2. * If a circle be described touching the axis major of an ellipse in one of the foci, and passing through one extremity of the axis minor, the semi-axis major will be a mean proportional between the diameter of the circle and the semi-axis minor.

3. * If AB, CD, two lines in an ellipse, not parallel to each other, make equal angles with either axis; the lines AC, BD, and AD, BC will also make equal angles with either axis.

4. If from points of the curve $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$, tangents be drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the chords of contact will be normal to the ellipse.

5. An oblique cone stands on a circular base; prove that $D \sec a$ is a third proportional to the axes of the section made by a plane passing through the centre of the base and perpendicular to the axis and a normal to the base.

6. A is the origin, B a point in the axis of y, BQ a line

parallel to the axis of x, in AQ produced if necessary, P is taken such that its ordinate is equal to BQ; shew that the locus of P is a parabola.

7. If a line be drawn through the centre of an ellipse, cutting the major axis at an angle θ, and the curve at an angle ϕ, prove that

$$(a^2 - b^2)\cos(2\theta - \phi) = (a^2 + b^2)\cos\phi.$$

8. Through a given point B of the axis of x a line is drawn parallel to the axis of y: to any point Q of this line another straight line is draw from the origin and produced to P until $QP = BQ$. Find the equation to the locus of P.

9. Let $P_1P_2P_3Q_1Q_2Q_3$ be six points lying in a conic section; let the areas of the triangles $P_1Q_2Q_3$, $P_1Q_3Q_1$, $P_1Q_1Q_2$, be denoted by $A_1B_1C_1$, and the areas of the triangles formed by putting P_2P_3 successively in the place of P_1 be denoted by A_2, B_2, C_2, A_3, B_3, C_3, respectively; then will

$$\frac{1}{A_1}\left(\frac{1}{B_2C_3} - \frac{1}{B_3C_2}\right) + \frac{1}{A_2}\left(\frac{1}{B_3C_1} - \frac{1}{B_1C_3}\right) + \frac{1}{A_3}\left(\frac{1}{B_1C_2} - \frac{1}{B_2C_1}\right) = 0.$$

10. The equations of three straight lines are

$$u\,(= x\sin\theta - y\cos\theta + c) = 0,\ u_1 = 0,\ u_2 = 0;$$

prove that the equations of the four circles, to each of which these lines are tangents, are

$$u^{\frac{1}{2}}\sin\frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}}\sin\frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}}\sin\frac{\theta_1 - \theta}{2} = 0,$$

$$u^{\frac{1}{2}}\sin\frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}}\cos\frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}}\cos\frac{\theta_1 - \theta}{2} = 0,$$

$$u^{\frac{1}{2}}\cos\frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}}\sin\frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}}\cos\frac{\theta_1 - \theta}{2} = 0,$$

$$u^{\frac{1}{2}}\cos\frac{\theta_2 - \theta_1}{2} + u_1^{\frac{1}{2}}\cos\frac{\theta - \theta_2}{2} + u_2^{\frac{1}{2}}\sin\frac{\theta_1 - \theta}{2} = 0.$$

DIFFERENTIAL CALCULUS.

1. If $\tan x = x + \frac{a_3 x^3}{1.2.3} + \frac{a_5 . x^5}{1.2.3.4.5} + \ldots$ shew that 1838

$$a_{2n+1} = \frac{2n.(2n-1)}{1.2} . 2a_{2n-1}$$

$$- \frac{2n(2n-1)(2n-2)(2n-3)}{1.2.3.4} . 2^3 a_{2n-3} + \&c.$$

2. Having given

$$c = \tan^2\theta \{(x + a \operatorname{cosect}\theta)^3 - x^2\},$$

determine θ so that $x^3 \tan^2\theta$ may be a maximum.

3. A point is taken within a triangle, such that the sum (σ) of its distances from the angular points of the triangle is the least possible: (s) is the sum of the sides of the triangle: α, β, γ are its angles and A its area. Shew that

$$s^2 - 2\sigma^2 = 4A (\operatorname{cosect}\alpha + \operatorname{cosect}\beta + \operatorname{cosect}\gamma - \sqrt{3}).$$

1. If $f(x)$ and $\phi(x)$ vanish when $x = a$, and (α), (β) be 1839 the indices of the lowest powers of (h) in the expansions of $f(a+h)$ and $\phi(a+h)$ respectively; prove that when $x = a$, $\frac{\log f(x)}{\log \phi(x)} = \frac{\alpha}{\beta}$.

2. Prove that, when (Q) and (P) are evanescent for any value of (x), $Q . \log P = 0$.

3. Prove the following formulæ:

(1) $d^{n-1}_{x=0}\left(\frac{x}{fx}\right)^n = d^n_{x=0}(f^{-1}x)$.

(2) $d^n_x (u_1 u_2 u_3 \ldots u_m) = (D_1 + D_2 + \ldots + D_m)^n u_1 u_2 \ldots u_m$,

the symbol $(D_r)^p$ being attached to u_r and being supposed to represent $d^p_x u_r$.

4. The evolute of a small portion of every plane curve near the point at which the curvature is a maximum or minimum is ultimately a portion of a semicubical parabola near the cusp.

5. Trace the curve whose equation is $r = a \cdot \cos\frac{\theta}{3}$; and shew that, if (A) be the point in which the curve meets the prime radius produced backwards, and $PSQR$ any chord drawn through the pole (S) meeting the curve in P, Q and R, the angles PAQ and QAR are each = 60°, and the angle ASQ equal to thrice the angle APS.

6. If any number of straight lines, drawn in any directions from one given point, have their lengths proportional to the cosines of the angles which they severally make with the longest line (a), and spheres be then described upon each of these lines as diameters: the surface, generated by the revolution of the curve whose equation, (taken from the given point as pole) is $r = a \cdot \left(\cos\frac{\theta}{2}\right)^2$, will envelope all these spheres.

1840 1. Shew that

$$\int_x \frac{(a + \beta\cos x)}{(a + b\cos x)^m}$$

$$= (-1)^{m-1} \frac{2}{1 \cdot 2 \cdot 3 \ldots (m-1)b} d_a^{m-1} \left\{\frac{ab - a\beta}{\sqrt{a^2 - b^2}} \right.$$

$$\left. \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)\right\}.$$

Deduce by this method $\int_x \frac{1}{(a + b\cos x)^2}$.

Also shew that

$$\int_x \frac{x^{m-1}}{(a + bx^n)^p} = \frac{(-1)^r}{(p-1)(p-2)\ldots(p-r)} d_a^r \int_x \frac{x^{m-1}}{(a + bx^n)^{p-r}}.$$

2. If $c^{n-2}y^2 = (x - a_1)(x - a_2)\ldots(x - a_n)$ be the equation to a curve, it cannot have maxima and minima ordinates for more than $\frac{n}{2}$ values of x, if n be even, nor for more than $\frac{n-1}{2}$ values, if n be odd.

1841 1. Eliminate by differentiation the constants and exponential forms from the equation $ae^y + be^{-y} = fe^x + ge^{-x}$.

2. If for any curve, r be the distance of any point in it

from a fixed point, and s the arc, then the perpendicular from the fixed point upon the normal $= r\frac{dr}{ds}$.

3. If ρ be the radius of curvature, then the radius of curvature ρ_1 of the evolute at the corresponding point $= \rho\frac{d\rho}{ds}$; and if ρ_n be the radius of curvature of the n^{th} evolute (the evolute of the first evolute being called the second evolute, and so on), then $\rho_n = \rho\frac{d\rho_{n-1}}{ds}$.

Prove these formulæ and give their geometrical interpretation.

4. Find the nature of the curve $y + 1 = 2x - x^2 \pm (2 - x)^{\frac{5}{2}}$ at the point corresponding to $x = 2$.

5. Shew that the curve whose equation is $\frac{y}{r} = \sin\frac{a}{r}$, where $y^2 - 2rx + x^2 = 0$, intersects the axis of x in an infinite number of points; also that the curve

$$y e^{\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)} = c\sqrt{x^2 - x + 1},$$

is one of those denoted by the equations

$$\log\left(\frac{y}{c}\right) = \int_x \frac{1}{x - \phi(x)} \text{ and } \phi^3(x) = x.$$

6. If θ be eliminated from the equations $\frac{dy}{dx} = \tan\theta$ and $\frac{d\theta}{dx} = \frac{ae^{c\theta}}{\cos\theta}$, shew that the curve represented by the resulting equation has this property, viz. the radius of curvature varies as the length of the arc.

1. Two ships are sailing uniformly with velocities u, v along lines inclined at an angle θ; shew that if a, b be their distances at one time from the point of intersection of the courses, the least distance of the ships is equal to 1843

$$\frac{(av - bu)\sin\theta}{(u^2 + v^2 - 2uv\cos\theta)^{\frac{1}{2}}}.$$

2. If $F\{x, y, z, a, b, c\} = \phi\{u, v, w, a, b, c\}$, where F is homogeneous of the n^{th} degree in x, y, z, and

$$u = \frac{dF}{dx},\ v = \frac{dF}{dy},\ w = \frac{dF}{dz},$$

prove that $x = (n-1)\frac{d\phi}{du}$, $y = (n-1)\frac{d\phi}{dv}$, $z = (n-1)\frac{d\phi}{dw}$.

3. Eliminate by differentiation the functions from

$$z = x^n f\left(\frac{y}{x}\right) + \frac{1}{y^n}\phi\left(\frac{x}{y}\right).$$

4. If u, v be functions of x, prove that

$$\frac{d^n(uv)}{dx^n} = v\frac{d^nu}{dx^n} + n\left(\frac{d}{dx}\right)^{n-1}\left(u\frac{dv}{dx}\right) - n\frac{(n-1)}{1\,.\,2}\left(\frac{d}{dx}\right)^{n-2}\left(u\frac{d^2v}{dx^2}\right) + \&c.$$

5. If $x = y e^y$, shew that

$$\sin(a+y) = \sin a + a_1\frac{x}{1} + a_2\frac{x^2}{1\,.\,2} + \ldots + a_n\frac{x^n}{1\,.\,2\ldots n} + \&c.$$

where $a_1 = \cos a$; and $a_n = (-1)^{n-1}(1+n^2)^{\frac{n-1}{2}}\cos\{a + (n-1)\cot^{-1}n\}$.

6. Find the locus of the ultimate intersections of perpendiculars drawn to normals to a parabola at the points where they cut the axes.

1844 **1.** Shew that

$$\frac{d^{2n}}{dx^{2n}}\left\{(a^2+x^2)^{n-\frac{1}{2}}\right\} = \left\{1\,.\,3\ldots(2n-1)\right\}^2 a^{2n}(a^2+x^2)^{-n-\frac{1}{2}}.$$

2. If a curve of the fourth order have three double points, the six tangents at those points all touch a conic section.

3. If (ρ) be the radius vector, (p) the perpendicular from the origin on the tangent, (s) the arc of a curve whether plane or of double curvature, $p\frac{dp}{d\rho} = \frac{p^2}{\rho} - \rho^2\frac{d^2\rho}{ds^2}$ and in a curve of double curvature, if (R) is the radius of absolute curvature, (ρ') the distance of the centre of absolute curvature from the origin, then

$$\rho^2\left(\frac{d\rho}{dp}\right) = \frac{2R^2p^2}{R^2+\rho^2-\rho'^2}.$$

In Finite differences, Prob. I. 1845

Shew that $r = a\cos 4\theta \,.\, e^{\theta}$ is the equation of such a spiral, and trace it.

The curve $A'B$ rolls on another curve AB, and A' is the point which comes in contact with A; prove that the curve traced out by A' in the neighbourhood of A is ultimately a semi-cubical parabola of which the parameter, (*i. e.* the constant a in the equation $ay^2 = x^3$) is equal to $\frac{9\rho\rho'(\rho-\rho')}{2(\rho-2\rho')^2}$, where ρ, ρ are the radii of curvature of AB, $A'B$ at the points A, A' respectively, reckoned positive when the concavities are turned the same way. 1846

1. Find the locus of a point, such that if from it a pair of tangents be drawn to an ellipse, the product of the perpendiculars dropped from the foci upon the line joining the points of contact shall be constant. 1847

Also determine the curve to which the chord of contact is always a tangent.

2. There are two systems of curves which cut each other everywhere at right-angles; OM, NP are two curves of one system, ON, MP are two of the other, and the arc $OM = x$, $ON = y$, $NP = \xi$, $MP = \eta$, and ρ, ρ' are the radii of curvature of PN, PM at the point P; prove that

$$\rho\frac{d^2\xi}{dx\,dy} = \rho'\frac{d^2\eta}{dx\,dy} = \frac{d\xi}{dx}\frac{d\eta}{dy},$$

the differential co-efficients being taken on the supposition that O remains fixed while P alters.

3. If $\phi f(x) = \phi F(x)$ for all values of x from a to b, and if c be a quantity not less than a nor greater than b such that $f(c) = F(c)$, and $f'(c)$, $F'(c)$ have opposite signs, prove that $\phi(y)$ is necessarily a maximum or minimum when $y = f(c)$.

4. Shew that there cannot be any curve such that if tan-

gents, PT, QT be drawn at any points P and Q to meet in T, the angle which PT subtends at a fixed point S may always bear a constant ratio to that subtended by QT, except when that ratio is one of equality, and prove that in this case the curve is a conic section having S for its focus.

1848 Find the equation to that involute of a cycloid which passes through the cusp, and shew that in the immediate neighbourhood of the cusp it becomes the curve $2a\,(4y)^3 = (3x)^4$, a being the radius of the generating circle.

1849 1. Find the locus of the ultimate intersections of the lines defined by the equation

$$x \cos 3\theta + y \sin 3\theta = a\,(\cos 2\theta)^{\frac{3}{2}},$$

where θ is the variable parameter.

2. If e be the eccentricity of a conic section, r the distance of any point from the focus, ρ the radius of curvature at that point, and ds an element of the arc of the curve, then

$$e^2 = \frac{dr^2}{ds^2} + \rho^2 \left(\frac{d^2r}{ds^2}\right)^2.$$

3. If u be a function of the independent variables x, y, z, given by the equations

$$u = f(s, t) \ldots\ldots\ldots\ldots (1),$$
$$s = F(lx + my + nz + kt)$$
$$= \phi(mz - ny) + \chi(nx - lz) + \psi(ly - mx),$$

and if $l^2 + m^2 + n^2 = k^2$; shew that

$$l\frac{du}{dx} + m\frac{du}{dy} + n\frac{du}{dz} + k\frac{du}{dt} = 0,$$

where $\frac{du}{dt}$ is obtained from (1) by considering s constant.

4. Trace the curve in geometry of two dimensions Prob. 8.

5. Given $y = xe^y$, prove that

$$\frac{y}{x} = 1 + \frac{2x}{\lfloor 2} + \frac{(3x)^2}{\lfloor 3} + \ldots\ldots + \frac{(nx)^{n-1}}{\lfloor n} + \ldots\ldots$$

INTEGRAL CALCULUS.

1. A GIVEN quantity of metal is to be made into a vessel in 1838
the form of a hollow cone of given thickness: determine the dimensions of the vessel that its content may be the greatest possible.

2. Two equal chords are drawn from the extremities of a diameter of a circle, and intersect in a point at a distance (c) from the center. Shew that the area of the surface generated by the revolution of the intercepted arc round a diameter through its middle point is equal to

$$2\pi a^2 \left\{\frac{(a-c)^2}{a^2+c^2}\right\},$$

where (a) is the radius of the circle.

Evacuate $\int_0^\infty \frac{(x^2+a^2)\,dx}{x^4+b^2x^2+b^4}$ and $\int_0^{\frac{1}{2}\pi} dx \cos(a \tan x)$. 1842

1. Shew that $\int_0^\infty \frac{dx}{x}(\cos ax - \cos bx) = \log\frac{b}{a}$, 1845

and that

$$\int_0^\infty \frac{dx}{x} \log\frac{1+2m\cos ax+m^2}{1+2m\cos bx+m^2} = \log(1+m)\log\frac{a^2}{b^2} \text{ or } \log\left(1+\frac{1}{m}\right)\log\frac{a^2}{b^2},$$

according as m is less or greater than unity.

2. Trace the locus of the equation

$$\frac{y}{b} = \int_0^\pi \log\left\{1 - 2\cos\theta e^{-\sin\frac{x}{a}} + e^{-2\sin\frac{x}{a}}\right\} d\theta.$$

3. Shew that the rectification of the curve of which the equation is $(x^2+y^2)^2 - 4ab(x^2+y^2) + 4a^2x^2 = 0$, where $(a-2b)(2a-b) = ab$, depends upon an elliptic function of the first order, of which the modulus is $\sqrt{5} = -2$.

4. If $F\left(x, \frac{1}{x}\right)$ be any symmetrical function of x and $\frac{1}{x}$, then will

$$\int_0^\infty \frac{dx}{x \,.\, F\left(x, \frac{1}{x}\right)} = 2\int_0^1 \frac{dx}{x \,.\, F\left(x, \frac{1}{x}\right)}.$$

1846 1. Trace the locus of the equation $y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x \cos u \, du}{\sqrt{x^2 + 2x \sin u + 1}}$, in which the sign of the square root is always taken so as to make the quantity in the denominator positive.

2. Compare the area of the curve in Integral Calculus, Prob. 9. with the area of the circle A.

1847 1. Find the limiting value of $x\epsilon^{-x^2}\int_0^x \epsilon^{x^2}\,dx$ when $x = \infty$.

2. Find the value of $\int_0^\infty \left\{\epsilon^{-(a+a\sqrt{-1})x} - \epsilon^{-(b+\beta\sqrt{-1})x}\right\}\frac{dx}{x}$, where a and b are positive, but a and β positive or negative; and shew that it is wholly real when $\frac{a}{a} = \frac{\beta}{b}$.

1848 1. Prove that the remainder after n terms of the infinite series $\frac{1}{1^a} + \frac{1}{2^a} + \frac{1}{3^a} \ldots$, where $a > 1$, lies between

$$\frac{1}{(a-1)\,(n+\frac{1}{2})^{a-1}}\left(= \int_{n+\frac{1}{2}}^\infty \frac{dx}{x^a}\right) \text{ and } \frac{1}{(a-1)\,(n+1)^{a-1}},$$

approaching much nearer to the former limit when n is large.

2. If $f(x)$ be positive and finite from $x = a$ to $x = a + h$, show how to find the limit of

$$\left\{f(a) \cdot f\left(a + \frac{1}{n}h\right) \ldots f\left(a + \frac{n-1}{n}h\right)\right\}^{\frac{1}{n}}$$

for $n = \infty$, and prove that the limit in question is less than $\frac{1}{h}\int_a^{a+h} f(x)\,dx$, assuming that the geometric mean of a finite number of positive quantities which are not all equal is less than the arithmetic.

Hence prove that $\epsilon^{\int_0^1 u dx} < \int_0^1 \epsilon^u dx$, unless u be constant from $x = 0$ to $x = 1$.

1849 1. Investigate the series

$$\frac{\theta^2}{4} = \frac{\pi^2}{12} - \cos\theta + \frac{\cos 2\theta}{2^2} - \frac{\cos 3\theta}{3^2} + \&c.$$

for values of θ between $-\pi$ and π.

2. In Geometry of two dimensions, Prob. 7. prove that $\int_0^\pi \phi d\theta = \frac{\pi^2}{2}$.

3. Find the whole area included between the curve in Geometry of two dimensions, Prob. 8, and the asymptote, and extend the geometrical description so as to include the whole of the curve given by the equation.

GEOMETRY OF THREE DIMENSIONS.

1. If S be a point, and EF, LM two straight lines given in 1838
position, and if a point P be taken such that PQ being drawn perpendicular to EF meeting LM in Q, PQ may bear an invariable ratio to SP, the locus of P will be a surface of the second order. Shew also that when SP, PQ are unequal, a section of the surface by a plane perpendicular to EF is a circle, and that when SP, PQ are equal, the surface has no center.

2. Two straight lines in space intersect in a point in a given plane. Determine the position of an eye to which their projections on that plane shall appear to include a given angle, and discuss the solution when the given angle is (1) zero, and (2) a right angle; shew also how to adjust the eye and the plane of projection that the lines may appear to be parallel.

3. A tangent plane is drawn at a given point of the surface $(x+y)^2 = az$: determine the volume of the space included between the surface, the tangent plane, and the planes of xz, and yz.

4. An elliptical area revolving round a diameter inclined to its major axis at a given angle, passes through a cylinder the axis of which is parallel to the diameter about which the area is revolving; determine the magnitude of the cylinder just sufficient to admit the area.

5. If $a\beta$, $a'\beta'$, $a''\beta''$, be respectively the semi-axes of the projections on the planes xy, xz, yz, of the intersections of an ellipsoid and concentric sphere. Shew that

$$\left(\frac{aa''}{a'}\right)^2 + \left(\frac{\beta\beta''}{\beta'}\right)^2 = 0.$$

6. The solid of which the surface is determined by the equation $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = a^{\frac{1}{2}}$ revolves round the fixed axis of (z), and makes for itself a cavity in a mass of yielding material: determine the form and magnitude of the cavity.

7. If ρ, ρ' be the greatest and least radii of curvature of a curve surface at a point, ϕ, ψ, the angles which a normal to the surface at the given point makes with the axes of x and y;

$$\text{shew that } \frac{1}{\rho} + \frac{1}{\rho'} = d_x \cos\phi + d_y \cos\psi.$$

8. A cone is revolving round its axis with a given angular velocity, when the length of the axis begins to be diminished gradually, and the vertical angle to be increased so that the volume of the cone remains unchanged: determine the angular velocity of the cone at the end of any time, and the number of revolutions it will make before the motion ceases.

1839 1. One thread of given length has its extremities attached to two given points A and B, whilst another thread of equal length has its extremities attached to the first point A and a third point C, and the two threads are passed through a small ring, which is moved so that they are both kept stretched in all positions: shew that the ring can never deviate from a certain plane, which is perpendicular to the plane ABC, and which cuts AB at a distance from A equal to

$$AB \cdot \frac{AB^2 - AC^2}{AB^2 + BC^2 - AC^2};$$

and find the nature, position and dimensions of its path.

2. If lines be drawn from the centre of an ellipsoid, (whose semi-axes are a, b, c,) parallel to the generating lines of an enveloping cone, the conical surface S so formed will intersect the ellipsoid in two planes parallel to the plane of contact. The locus of the vertex of the enveloping cone which causes one of the planes to coincide with the plane of contact is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 2;$$

and the locus of the vertices of all cones formed in the same manner as S which intersect the ellipsoid in plane curves, is the straight line joining the centre of the ellipsoid and the vertex of the enveloping cone.

3. At the vertices of an ellipsoid an elliptic paraboloid may be found which at its vertex will have a complete contact of the *third* order with the surface, and *every* plane section, passing through the vertex of the ellipsoid, will have a contact of the third order with the corresponding section of the paraboloid.

✓ 1. Find the equation to the surface generated by a line 1840
which always passes through each of two given lines in space, and also through the circumference of a circle whose plane is parallel to them both, and whose centre bisects the shortest distance between them.

2. If the axes of two equal cylinders of radius a, intersect at an angle α, the volume common to both is $= \frac{16}{3} \frac{a^3}{\sin \alpha}$, and the surface of each intercepted by the other is $= 16 \frac{a^2}{\sin \alpha}$.

3. A conical surface will intersect an ellipsoid in two plane curves or none; and when the vertex of the surface is fixed, the intersections of the two plane sections will all lie in one plane.

✓ 4. The sum of the squares of the projections of any three conjugate diameters of an ellipsoid (whose semi-axes are a, b, c,) upon a given principal diameter is constant; and the tangent planes at the extremities of the three conjugate diameters intersect in an ellipsoid whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3.$$

5. Two points A, B in a surface of the second order are joined and produced to C; from any point P in the surface, draw AP meeting the plane of contact (of the enveloping cone whose vertex is C) in the point Q; the locus of the intersection of CP, BQ will be a surface of the second order.

6. If tangent planes be drawn at the extremities of any chord to a surface of the second order, their intersection lies in the plane diametral to that chord; and if the chord be supposed always to pass through a given point, the locus of the intersections of the pairs of tangent planes will be a plane; also when the given point is external, this locus is the plane of contact of all tangent planes which pass through the point.

7. A plane passing through the centre of an ellipsoid is inclined to the plane of xy at an angle i, and its trace on the plane of xy is inclined to the axis of x at an angle ϕ; $2a'$, $2b'$ are the axes of the elliptic section, and a the angle which its axis major makes with the trace of the cutting plane on the plane of xy. Prove that $\tan 2a = \frac{2C}{A-B}$,

$\frac{1}{a'^2} = \frac{A+B-\sqrt{(A-B)^2+4C^2}}{2}$, and $\frac{1}{b'^2} = \frac{A+B+\sqrt{(A-B)^2+4C^2}}{2}$,

where A, B, C, have the values given in Prob. 1. Plane Trigonometry.

8. Prove that the Stereographic projection of the sphere gives a representation which is similar to the part represented, immediately about any assigned point, but that the scale (i. e. the ratio of any small line on the sphere to its projection); varies from point to point, and is $= \frac{1}{2}\sec^2\frac{\theta}{2}$, where θ is the angle subtended at the centre of the sphere, by the arc of a great circle drawn from the part represented to that pole of the primitive plane which is opposite to the eye.

Also shew that if ϕ be the angle made by a plane through each point to be represented and the axis, with a fixed plane through the axis, and if upon any plane a series of points be taken whose rectangular co-ordinates are

$$x = a\left(\tan\frac{\theta}{2}\right)^{\lambda}\cos\lambda\phi,\ y = a\left(\tan\frac{\theta}{2}\right)^{\lambda}\sin\lambda\phi,$$

where a and λ are any constants, a projection will be

made possessing the above property, the scale being now $= \frac{a\lambda}{a} \frac{\left(\tan\frac{\theta}{2}\right)^{\lambda}}{\sin\theta}$, where a is the radius of the sphere.

1. A cloud, or other object known to be moving in a horizontal plane, is observed by a person within a room; from the line in which it seems to move across the window and the position of the eye, determine the direction of its motion. 1841

2. A conical surface, whose vertex is A, is cut by a plane which is parallel to one of its slant sides AC, and the part of the surface which contains the vertex is developed into a plane, the surface being again cut along AC; shew that the polar equation to the curved boundary of the plane figure, measuring from A, is $r = \frac{2AB}{1 + \cos(\theta \operatorname{cosec} a)}$, where $2a$ is the vertical angle of the cone, and B the point of intersection of the cutting plane with the slant side which is opposite to AC.

3. A plane is moved so as to cut off from the co-ordinate planes areas whose sum is always equal to $2n^2$; shew that the surface, to which the plane is always a tangent plane, is represented by the equation

$$(3x - u)(3y - u)(3z - u) + 9n^2u = 0,$$

where $u = x + y + z \mp \sqrt{3n^2 + x^2 + y^2 + z^2 - xy - xz - yz}$.

4. From each point of the exterior of two concentric ellipsoids, whose axes are in the same directions, tangent planes are drawn to the surface of the interior one; shew that all the planes of contact, corresponding to the several points of the exterior surface, touch another concentric ellipsoid.

Shew also, that if tangents be drawn from each point of any plane curve of the second order to any other curve of the second order in the same plane, however situated, the lines which join the points of contact of the pairs of tangents drawn from each point of the first curve, touch another curve of the second order.

1842 ✓ 1. If two straight lines at right angles to each other on an inclined stratum make angles α, β with the horizontal plane, the dip (θ) of the stratum is given by the equation $\cos\theta = \{\cos(\alpha+\beta)\cos(\alpha-\beta)\}^{\frac{1}{2}}$.

✓ 2. If a normal to the surface, of which the equation is

$$x\cos nz - y\sin nz = 0,$$

move along any one of its generating lines, it will trace out a hyperbolic paraboloid.

3. If a, b, c be the semi-axes of an ellipsoid taken in order, and e, e' the eccentricities of the principal sections containing the mean axis, shew that the perpendiculars from the centre on the tangent planes at every point of the section of the surface made by the plane $abe'x = c^2ez$ are equal.

✓ 4. What is the geometrical interpretation of the equation $(x-a)^2(y-b)^2 + (x-a)^2(z-c)^2 + (y-b)^2(z-c)^2 = 0$?

5. If four similar surfaces of the second order intersect each other, the planes of their intersections two and two all pass through one point.

6. Among the surfaces included in the equation

$$x\frac{dz}{dx} + y\frac{dz}{dy} = 0,$$

find that in which the principal radii of curvature are equal but of opposite signs.

1843 1. A circle with its plane vertical rolls along a given curve on a horizontal plane: if x_1y_1 be the co-ordinates of the point of contact of the circle and curve, and xyz the co-ordinates of any fixed point P on the circle, shew that

$(x-x_1)\,dx + (y-y_1)\,dy + z\,dz = 0$, $(x-x_1)^2 + (y-y_1)^2 + z^2 - 2az = 0$,

a being the radius of the circle. Hence shew how to find the equation to the curve generated by P, and take for example the common cycloid.

2. The circular base of a cone remains constant, while the vertex moves in a segment of a circle described on a diameter of the base, having its plane at right angles to the base; find

the surface on which lie all the sub-contrary sections of the cone.

3. A sphere described from the origin with radius $\frac{abc}{ac+ab+bc}$ will touch the surface $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1$ in points which are *umbilici*.

4. Ellipsoids defined by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, whose axes satisfy the condition $\frac{k^2}{a^2}+\frac{l^2}{b^2}=1$, all touch a certain straight line in the same point; and if the straight line meet the plane of xy in the point (a', b'), the co-ordinates of the projection of the point of contact on that plane are $\left(\frac{k^2}{a'}, \frac{l^2}{b'}\right)$.

5. An ellipsoid revolves about a line through its centre in the plane of (ab) inclined at an angle θ to (a): if $e^2 = 1-\frac{c^2}{a^2}$, $e'^2 = 1-\frac{c^2}{b^2}$ the spheroid which envelopes it in every position is

$$e^2e'^2(x+y\tan\theta)^2=(x^2+y^2+z^2-c^2)(e^2+e'^2\tan^2\theta).$$

6. From every point of the surface

$$(1)\quad 4z(lx^2+l'y^2)+ll'(x^2+y^2)=0$$

as vertex, enveloping cones are drawn to the paraboloid

$$(2)\quad ll'z-(lx^2+l'y^2)=0;$$

shew that the planes of contact all touch the surface

$$4z(lx^2+l'y^2)-(l^2x^2+l'^2y^2)=0:$$

also shew that the surface (1) contains the directrices of all the sections of (2) made by planes through its axis.

7. Having given the radii of the outer and inner shaft of a winding staircase, and its constant inclination; find the length of the longest beam that can be carried up it.

8. Conceive a system of surfaces of such a nature, that a normal at any point of one of them is a normal to all the rest; let $u=0$ be the equation to one of these surfaces, let the normal at any point P be the axis of z, and let the distance of the origin

from the point P be a harmonic mean between the greatest and least radii of curvature at that point. Then shew that at the point P

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = \frac{1}{z}\frac{d^2(uz)}{dz^2}.$$

1844 1. A particle attracted to a fixed centre of force (the force varying as the distance) is projected with a given velocity. If the locus of the direction of projection is a plane, the locus of the orbit is an ellipsoid of which the particle's original position is an umbilical point.

2. A sphere (C) and a plane are given; shew that, if any sphere (C') be described, touching the plane at a given point, and cutting C, the plane of section always contains a given right line. Shew also that, if the point of contact be not given, and if the plane of section always contain a given point, the centre of the sphere C' will always be upon a given paraboloid.

3. The equations of four planes are

$$\alpha x + \beta y + \gamma z = 1 \ldots(1), \qquad \alpha x + \beta' y + \gamma z = 1 \ldots(3),$$

$$\alpha' x + \beta y + \gamma z = 1 \ldots(2), \qquad \alpha x + \beta y + \gamma' z = 1 \ldots(4);$$

$\alpha, \beta, \gamma, \alpha', \beta'$, and γ' being variable parameters: (1) always contains a given right line, and (2), (3), (4) always contain three given points respectively: shew that, when the given right line and the three given points lie in the same plane, the locus of the point of intersection of (2), (3) and (4) will be a right line passing through the origin.

Shew hence that, if ABC, $A'B'C'$ be two triangles in the same plane, and if the intersections of AB and $A'B'$, BC and $B'C'$, CA and $C'A'$, respectively, be in the same right line; then the lines joining A and A', B and B', C and C', respectively, meet in the same point.

4. Of a series of cones enveloping an ellipsoid, the vertices lie on a concentric ellipsoid, similar to the given one and similarly situated. Prove that any two cones of the series intersect one another in two planes.

5. Prove that if, in the last question, the vertices are supposed to lie also on a third ellipsoid, concentric with the other two, and similarly situated, and whose axes are respectively as the squares of theirs, these two planes are at right-angles to one another.

6. If the normal plane of a curve constantly touches a given sphere, the curve is rectifiable.

7. Each pair of three surfaces of the second order have two plane curves in common, and one of these plane curves is common to all three surfaces. Shew that the planes of the other three sections intersect in one straight line.

1. Having given the equations of a curve in space referred 1845
to three rectangular axes, find the length of a perpendicular from the origin upon the tangent at any point.

Ex. $x = a\cos\theta, y = a\sin\theta, z = \frac{a}{2}(e^{\theta} + e^{-\theta})$.

. Prove that if the perpendicular be invariable the involute lies on the surface of a sphere.

2. An ellipsoid is cut by a plane, the distance of which from the centre bears a constant ratio to its distance from the parallel tangent plane. Shew that the volume of the cone whose base is the section and vertex the centre of the ellipsoid is invariable.

3. A plane is drawn according to a certain assigned law cutting an ellipsoid; find the locus of the vertex of the cone which touches the ellipsoid in the curve of intersection.

If a, b, c be the semi-axes of the ellipsoid, $\frac{x}{a} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$ the equation to the cutting plane, α, β, γ being connected by the relation

$$\frac{a^4}{\alpha^2} + \frac{b^4}{\beta^2} + \frac{c^4}{\gamma^2} = \text{constant},$$

the locus will be a sphere.

4. A sphere and ellipsoid which intersect are described about the same point as centre; prove that the product of the

areas of the greatest and least sections of the ellipsoid, made by planes passing through the centre and any point in the line of intersection of the two surfaces, will be constant.

5. If from any point tangent planes are drawn to a surface of the n^{th} order, the points of contact will all lie on a surface of the $(n-1)^{\text{th}}$ order.

6. All sections of the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, which are at the same distance p from its centre, have their centres in the surface of which the equation is

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2=p^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right).$$

1846 1. Find the volume of the solid the equation to the surface of which is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^4}{c^4}=1$.

2. If P be any point in a surface, a, b the principal radii of curvature at P, r the radius of curvature of the normal section made by a plane inclined at an angle θ to the principal section to which a refers, $PQ=s$ an indefinitely small arc in this section, prove that if D be the minimum distance of the normals at P and Q, c the distance from P of their point of nearest approach,

$$D^2=\frac{\sin^2\theta\cos^2\theta\,(a-b)^2s^2}{b^2\cos^2\theta+a^2\sin^2\theta},\qquad c=\frac{ab\,(b\cos^2\theta+a\sin^2\theta)}{b^2\cos^2\theta+a^2\sin^2\theta}.$$

Express also $\left(\frac{D}{s}\right)^2$ and c in terms of a, b and r.

3. The equations to a system of lines in space, straight or curved, contain two arbitrary parameters; show how to find whether the lines can be cut at right angles by a system of surfaces, and when they can, show how to find the equation to that system.

Examples. (1) Let the lines be a system of right lines each of which intersects two given right lines which are perpendicular to each other but do not intersect. (2), Let the equations to the system of lines be $Ax^\alpha=By^\beta=Cz^\gamma$, where A, B and C are arbitrary.

4. A straight inelastic band is wrapped smoothly on the surface of a cone; shew that however long it may be, the two ends of either of its edges cannot be made to meet, if the vertical angle of the cone be greater than 60°.

5. Three forces act along three lines which may be considered as generating lines, in the same system, of a hyperboloid of one sheet; shew that if the forces admit of a single resultant, it must act along another generating line of the same system.

6. An ellipsoid is constantly touched in two points by a sphere of given radius whose centre moves in one of its principal planes; find the locus of the centre of the sphere, and shew that it will describe a portion of an equilateral hyperbola, if the plane in which it moves be that which contains the greatest and the least axes, and if the foci of the other principal sections be equidistant from the centre of the ellipsoid.

7. Find the locus of the ultimate intersections of a series of planes touching two parabolas which lie in planes perpendicular to each other, and have a common vertex and axis.

8. The equation to a system of surfaces contains one arbitrary parameter; normals are drawn at all points of one of these surfaces, and the lengths of the normals are taken proportional to the ultimate distances between the surface in question and a consecutive surface; shew how to find the equation to the locus of the extremities of the normals.

9. Assuming the curvature at any point of a surface to be measured by the product of the reciprocals of the principal radii, shew that the lines of equal curvature on the surface of an elliptic paraboloid are projected on a plane perpendicular to its axis in similar ellipses whose axes are proportional to the parameters of the principal sections of the paraboloid.

1. Apply this property to prove that if PQ, PR be any 1847
two chords of an ellipsoid, in the same plane with, and inclined at the same angle to a fixed chord PK, then the locus of all

the possible intersections of PQ, $P'Q'$, &c. will be the line PK and a plane curve.

2. At the point in which the surface

$$\left(\frac{x^2}{a^2} - 1\right)^2 + \left(\frac{y^2}{b^2} - 1\right)^2 = \frac{z^2}{c^2} + 1$$

meets the axis of z, an elliptic paraboloid may be found, which has, at its vertex, a complete contact of the third order with the surface.

3. If R be the radius of absolute curvature at any point of a curve defined by the intersection of two surfaces $u_1 = 0$, $u_2 = 0$, and r_1, r_2 be the radii of curvature of the sections of $u_1 = 0$, $u_2 = 0$ made by the tangent planes to $u_2 = 0$, $u_1 = 0$ respectively at that point, prove that R, r_1, r_2 will be connected by the relation

$$\frac{1}{R^2} = \frac{1}{r_1^2} - \frac{2\cos\theta}{r_1 r_2} + \frac{1}{r_2^2},$$

θ being the angle between the tangent planes.

4. Tangent planes to the surface whose equation, referred to rectangular co-ordinates, is $\frac{x^3}{a^3} + \frac{y^3}{b^3} + \frac{z^3}{c^3} = 1$ pass through a point P; prove that a sphere can be described through the curve of contact provided P lie in a certain right line passing through the origin.

5. The equations to a system of right lines in space contain two arbitrary parameters; prove that when the roots of a certain quadratic are real and unequal, there are two planes passing through a given line of the system which contain consecutive lines.

6. The radius vector of any point in the surface of a solid differs from a constant by a small quantity of the order a, and the angle which it makes with the normal is of the same order; prove that if small quantities of the order a^2 are neglected the sphere which has the same volume as the given solid is also that which has the same surface.

7. A regular octahedron is inscribed in a cube, so that the corners of the octahedron are in the centres of the faces of the

cube; prove that the volume of the cube is six times that of the octahedron.

8. A sphere touches each of two right lines which are inclined to each other at a right angle, but do not intersect; prove that the locus of its centre is a hyperbolic paraboloid.

9. A circle always touches the axis of z at the origin, and passes through a fixed straight line in the plane of xy; find the equation to the surface generated. Shew that the origin is a singular point, and that in its immediate neighbourhood the surface may be conceived to be generated by a circle having its plane parallel to that of xy, and its radius proportional to z^2.

10. P is a point in the base of a tetrahedron $VABC$, of which V is the vertex, so taken that the volume of the parallelopiped constructed on VP as diagonal, and having three of its edges coincident with VA, VB, VC, is a maximum; supposing the base of the tetrahedron to turn in all directions round a given point, while the edges adjacent to V remain fixed in position, find the locus of P.

11. A sphere touches an elliptic paraboloid at the vertex, and has its diameter a mean proportional between the parameters of the principal sections of the paraboloid; supposing the curve of intersection of the sphere and paraboloid to be projected on the tangent plane at the vertex, find the area of the curve of projection.

1. If three chords be drawn mutually at right angles through a fixed point within a surface of the second order whose equation is $u = 0$, shew that $\Sigma \frac{1}{Rr}$ will be constant, where R and r are the two portions into which any one of the chords drawn through the fixed point is divided by that point. 1848

Prove also that the same will be true if instead of the fixed point there be substituted any point in the surface whose equation is $u = c$.

2. Find the locus of the foot of the perpendicular let fall from the origin on the tangent plane to the surface $xyz = a^3$;

point out the general form of the required surface, and find the whole included volume.

3. A parallelopiped is constructed by drawing tangent planes at the extremities of three conjugate diameters of an ellipsoid; prove that the diagonals of the parallelopiped form a second system of conjugate diameters, and that the relation between the two systems is reciprocal.

4. A plane moves so as always to enclose between itself and a given surface S a constant volume; prove that the envelope of the system of such planes is the same as the locus of the centres of gravity of the portions of the plane comprised within S.

5. If O be a given point in a surface of the second order, and OA, OB, OC any three chords passing through O mutually at right angles, shew that the plane ABC will always pass through a fixed point.

1849 1. If planes be drawn through any two generating lines of a hyperboloid which intersect, shew that they will cut the surface in another pair of generating lines.

2. If $u = f(x, y, z)$ be a rational function of x, y, z, and if $u = 0$ be the equation to a surface, for a point $(a, b, c,)$ of which, all the partial differential co-efficients of u as far as those of the $(n-1)^{\text{th}}$ order vanish, shew that the conical surface whose equation is

$$\left\{(x-a)\frac{d}{da} + (y-b)\frac{d}{db} + (z-c)\frac{d}{dc}\right\}^n f(a, b, c) = 0$$

will touch the proposed surface at the point a, b, c.

3. A rod AB is fixed to a universal joint at A, and another rod BP is connected to it by a universal joint at B: all directions of either rod being equally probable, find the chance of P lying between two spherical surfaces of given radii, whose common centre is A; and shew that the chance of P lying within a given elementary portion of space containing the point P varies inversely as AP_1.

4. Determine the form of the termination of a honey-comb cell, on this principle, that if a sphere which will just pass through the hexagonal transverse section be dropped into the cell, the unoccupied space at the extremity of the cell shall be the least possible.

5. Determine the condition to which the vertices of a system of cones which envelope an ellipsoid must be subject, in order that the centres of the ellipses of contact may be equidistant from the centre of the ellipsoid.

6. The tangent plane to a surface S cuts an ellipsoid, and the locus of the vertex of the cone which touches the ellipsoid in the curve of intersection is another surface S'. Prove that S and S' are reciprocal, that is, that S may be generated from S' in the same manner as S' has been generated from S.

Find a curve, such that if perpendiculars be drawn from 1845
two given points upon the tangent at any point, the area included by the perpendiculars the tangent and the distance between the points shall be a maximum.

Assuming that $\sin x + \frac{\cos x}{x}$ is a particular integral of the 1848
equation $d_x^2 y + \left(1 - \frac{2}{x^2}\right) y = 0$, find the complete integral of the equation $d_x y^2 + \left(1 - \frac{2}{x^2}\right) y = x^2$.

A curve is defined by this property, that the radius of cur- 1849
vature at any point is a given multiple of the portion of the normal intercepted between the point and the axis of abscissæ; prove that the length of any portion of the curve may be expressed in finite terms of the ordinates of its extremities.

DIFFERENTIAL EQUATIONS.

1. The locus of the intersection of a tangent to a curve and 1838
a perpendicular to the tangent drawn from a given point, is an ellipse having the given point in one extremity of its major axis: find the equation to the curve.

2. If $R = (\int_{r=1} - \int_{r=0}) \sqrt{(1 - x^2)} \cos rx$,
then $r d_r^2 R + 3 d_r R + r R = 0$.

3. Determine the nature of the curve in which if (θ) be the angle which the curve makes with the axis of (x) at any point, (s) the length of the arc between the origin and that point, and (a, a_1) constants,

$$\tan\theta \cdot \tan\left(\frac{s}{a}\right) = a_1.$$

1839 1. Having given r particular solutions of a general linear equation of differential coefficients, or finite differences, shew that it may be depressed r orders; and having given one particular integral, determine the first reducing equation in each case.

2. If m be the greatest root of the equation $m^2 + m = a$,

$$C d_{r=n}^2 \left\{\frac{\cos(x\sqrt{r} + a)}{x^m \sqrt{r}}\right\} \text{ or } C x^{m+1} (\int_{r=n} - \int_{r=-n}) (r^2 - n^2)^m \cos(rx + a)$$

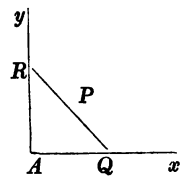
are general values of y in the equation $d_x^2 y + \left(n^2 - \frac{a}{x^2}\right) y = 0$ according as m is an integer or fraction: and in the first case $(d_x^2 + n^2)^{m+1} u = 0$ where $u = y x^m$; apply the first or third result to solve the equation

$$d_x^2 y + \left(n^2 - \frac{6}{x^2}\right) y = 0.$$

3. A ship (A) sailing right before the wind with a velocity (u) is chased by another ship (B) which always steers directly towards (A) and moves with a velocity $u(1 + \cos\theta)$, where θ is the angle which the direction of B's motion makes with that of the wind; when the chase begins, the distance between the ships is (a), and the line joining them is perpendicular to the direction of the wind; shew that B's path will be a quadrant of a circle, and that it will overtake A in the time $\frac{a}{u}$.

1841 1. Prove that $\Delta_x^m \Delta_y^n u_{x,y} = \Delta_y^n \Delta_x^m u_{x,y}$, and that each expression $= (u_{'} - 1)^m (u_{''} - 1)^n$, if generally $u_{'}^r u_{''}^s = u_{x+rh,\, y+sk}$, where h and k are the increments of x and y respectively.

2. In any straight line QR, which cuts the rectangular axes AQx, ARy in Q and R, a point P is taken such that $PQ \div PR$ is always constant. Find the curve which always touches QR at the point P; explain the meaning of the arbitrary constant introduced; and shew that, for the same constants, there are two curves which answer the conditions of the problem.



1. Prove that the orthogonal trajectory of a series of ellipses described round the same foci is a confocal hyperbola. 1842

2. If $$n^2u + (2n+1)\,a\,\frac{du}{da} + (a^2 - \beta^2)\,\frac{d^2u}{da^2} = 0,$$

shew that the equation is satisfied by

$$u = \int_0^\pi \frac{dx}{(a + \beta\cos x)^n}.$$

3. Integrate the equations

$$z\,\frac{d^2z}{dx\,dy} + a\,\frac{dz}{dx}\,\frac{dz}{dy} = 0,$$

$$xy\,\frac{d^2z}{dx\,dy} - x\,\frac{dz}{dx} - y\,\frac{dz}{dy} + z = xy;$$

SP, SP' are two radii vectores of a curve, SY, SY' are corresponding perpendiculars from the origin on the tangents at P, P'. Find the curve when if $SP = SY'$ then shall also $SP' = SY$. 1843

1. Prove that the equations 1844

$$\frac{d^ny}{dx^n} + p_1\,\frac{d^{n-1}}{dx^{n-1}}\,y + \&c. + p_ny = 0,$$

$$\frac{d^ny}{dx^n} - \frac{d^{n-1}y}{dx^{n-1}}\,(p_1y) + \&c. \pm p_ny = 0,$$

where p_1, p_2, &c. may be any functions of x, are so related to one another that any value of y which satisfies either is an integrating factor of the other.

2. Determine a curve which has the following property. If any series of points be taken on it so related that the tangent

at any one of them meets the ordinate to the next in the axis of x, then the corresponding subtangents will be in geometrical progression.

3. Of $$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = 0,$$

a solution is

$$v = \iiint \frac{dx'dy'dz'}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{1}{2}}} :$$

Shew there is a solution, similar in form, of

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + \frac{d^2v}{dw^2} + \&c., = 0,$$

whatever be the number of variables.

DEFINITE INTEGRALS.

1839 If $\phi(a-x)$ be an algebraical function which vanishes when $x = c$, prove that

$$\int_0^c dx \phi(a-x) \cdot x^{n-1} = \Gamma(n) \left(\frac{d}{da}\right)^{-n} \phi(a).$$

1840 1. Find $\int_0^\infty \frac{x^{m-1}}{(1+x^n)^r}$, and thence shew that

$$\left\{\int_0^\infty x^{m-1} e^{-x^n}\right\} \left\{\int_0^\infty x^{n-m-1} e^{-x^n}\right\} = \frac{\pi}{n^2 \sin\frac{m\pi}{n}}.$$

2. Prove the formula

$$\int_{-\infty}^{\infty} t^n e^{-t^2} = \frac{\sqrt{\pi}}{2^n} d_{i=0}^n e^{i^2}.$$

1844 1. Find the values of the integrals

$$\int_0^\infty dx \int_0^\infty dy \ldots \cos(ax + by + \ldots),$$

and $$\int_0^\infty dx \int_0^\infty dy \ldots \sin(ax + by + \ldots),$$

the number of variables in each being n.

2. Find the value of

$$\int_0^\infty \{e^{-\frac{a^2}{x^2}} - e^{-\frac{b^2}{x^2}}\} dx,$$

and apply Wallis's theorem to shew that if n increase *sine limite* $\sqrt{n}\int_0^{\frac{\pi}{2}} \cos^{2n}\theta\, d\theta$, and $\sqrt{n}\int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta\, d\theta$ are both ultimately equal to $\frac{\sqrt{\pi}}{2}$.

3. A chemical analysis has resolved a compound substance into two elements. There was a loss l in the operation, so that the sum of the quantities obtained for the elements is less by l than the whole quantity of the compound substance. If the probability of an error ϵ in the determination of each element be $\frac{h}{\sqrt{\pi}} e^{-h^2\epsilon^2} d\epsilon$, find the most probable value of h.

1. Prove that $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta = 2\pi$, 1845

and that $\int_0^{\frac{\pi}{2}} \frac{\sqrt{1-c}}{1-c\cos^m\theta} \cdot d\theta = \frac{\pi}{\sqrt{2m}}$,

when c is indefinitely nearly equal to 1, m being a positive quantity.

2. The value of the definite integral $\int_0^{\frac{\pi}{2}} \log(1 + n\cos^2\theta)\, d\theta$ may be found, whatever positive value is given to n, from the formula

$$\int_0^{\frac{\pi}{2}} \log(1 + n\cos^2\theta)\, d\theta = \frac{\pi}{4} \log\{(1+n)(1+n_1)(1+n_2)\ldots\},$$

where $n, n_1, n_2\ldots$ are quantities connected by the equation

$$n_{r+1} = \frac{n_r^2}{4(n_r+1)}.$$

3. If $F(x)$ be an algebraical polynomial of less than n dimensions,

$$\int_b^a \frac{F(x)\, dx}{(x-c)^n} = \frac{1}{\lfloor n-1} \frac{d^{n-1}}{dc^{n-1}} \left\{F(c) \log\frac{a-c}{b-c}\right\}.$$

If $u_n = \int_0^{\pi} \frac{d\theta}{(1 + e\cos\theta)^n}$, where n is a positive integer, and 1846 $e^2 < 1$, prove that

$$u_n = \frac{\pi}{1 \cdot 2 \ldots (n-1)} \left(n - 1 + \frac{d}{d\lambda}\right)\left(n - 2 + \frac{d}{d\lambda}\right) \ldots \left(1 + \frac{d}{d\lambda}\right) (1 - \epsilon^{2\lambda})^{-\frac{1}{2}}$$

$$u_n = \frac{(-1)^{n-1} \pi f^n}{1 \cdot 2 \ldots (n-1)} \frac{d^{n-1} (f^2 - 1)^{-\frac{1}{2}}}{df^{n-1}},$$

where $\lambda = \log e$, and $f = \frac{1}{e}$.

1849 1. Shew that

$$\int_0^\infty \frac{\sqrt{x} \cdot \log x \cdot dx}{(1+x)^2} = \pi.$$

2. Prove that

$$\int_0^\pi \frac{2e + (1+e^2)\cos\theta}{(1 + 2e\cos\theta + e^2)} \log_e (1 + 2e\cos\theta + e^2)\, d\theta = \pm \pi \frac{e^{\times 1}}{1 - e^2},$$

the upper or lower sign being taken according as e is less or greater than unity.

3. Shew that

$$\int_0^\infty \log x \cdot \log\left(\frac{x^2 + a^2}{x^2}\right) \cdot dx = \pi a (\log a - 1).$$

FINITE DIFFERENCES.

1838 1. FIND the x^{th} term and sum of x terms of 1, 2, 3, 8, 13, 30.....

2. Copies of a manuscript are multiplied in the following manner. The original is given to a person who remains one day idle, makes three copies on the second day and afterwards four copies daily. At the end of each day the copies made during the day are given each to a different person who remains one day idle, makes three copies on the second day and afterwards four copies daily. In like manner the copies thus obtained are given to others who work at the same rate. Find the number of copies existing at the end of x days.

1839 (1) Prove the following formula for transforming the independent variable from x to ϵ^s, and apply the result to the case in which $n = 5$.

$d_x^n y = \frac{\Delta 0^n}{1} u_1 + \frac{\Delta^2 0^n}{1.2} u_2 + \ldots + \frac{\Delta^n 0^n}{1.2 \ldots n} u_n$, where $u_r = t^r d_t^r y$ and $t = \epsilon^x$.

(2) $d^n (1 \epsilon^t) \left(\frac{1}{1-t}\right)$ is the generating function of x^n.

1. Prove the following formula 1840

$$\Delta_{=0}^{r} x^n = r \Delta_{=1}^{r} x^{n-1};$$

2. If $G(X) = \phi t$ represents the generating function of X, then

$$\frac{\phi t}{a - \log_\epsilon t} = G(\epsilon^{-ax} \int_x \epsilon^{ax} X), \text{ and } \frac{t \phi t}{1 - at} = G\left(a^{x-1} \Sigma \frac{X}{a^x}\right).$$

Apply the last two formulæ to solve by the calculus of generating functions the linear differential equation or equation of differences of the n^{th} order when all the co-efficients are constant, and the second member of the equation = X.

3. Two radii SP, SQ of a curve are drawn so as to include a given angle; find its nature when the inclination of the tangents is constant; and shew from the result that a parabola having S for its focus is a particular case.

4. Find the nature of the curve in which every chord subtending a given angle α at a fixed point A shall also subtend a given angle β at a fixed point B.

Shew also that $\rho \cos m\theta = a \cos(m \pm 1)\theta$ is a particular solution, where $AB = a$, and m is a constant depending on the values of α and β.

1. Each of a series of numbers beginning from unity is 1841 formed by multiplying the preceding one by a, and adding $2m$ or subtracting m according as the order of the number to be formed is odd or even: shew that the x^{th} number in the series is equal to

$$\frac{a^2 - 1 - m(a-2)}{a^2 - 1} \cdot a^{x-1} - \frac{m}{2}\left\{\frac{1}{a-1} + \frac{3(-1)^x}{a+1}\right\}.$$

2. A seed grain is planted, and when one year old produces the next year tenfold; and when two years old and upwards produces annually eighteenfold. Every grain is planted as soon

as it is produced, and the produce of each is according to the above law. Find the number of grains at the end of the x^{th} year.

3. Integrate the equation, $u_{x+2} - 2au_{x+1} + a^2u_x = a^x$.

1843 1. The sum of the (m) least roots of the equation $x^m - 1 = \epsilon^x$ is a series of which the $(r)^{\text{th}}$ term is $\frac{1}{r}\frac{\Delta^r 0^{rm-1}}{1\,.\,2\ldots(rm-1)}$.

2. If $G(u_x)$ be the generating function of u_x,

$$G(\log_\epsilon x) = \frac{\log_\epsilon(1-t)}{\log_\epsilon(t)}, \quad G(\cos x\phi) = \frac{t\cos\phi - t^2}{1 - 2t\cos\phi + t^2},$$

$$G\tan^{-1}x = \frac{1}{2\log_\epsilon\left(\frac{1}{t}\right)}\int\left(\frac{t^{\nu(-1)}}{t}\int\frac{t^{-\nu(-1)}}{1-t}\,dt + \frac{t^{-\nu(-1)}}{t}\int\frac{t^{\nu(-1)}}{1-t}\,dt\right)dt.$$

1845 Find the equation of a family of spirals, which shall have this property, that the ratio of any two radii vectores at right angles to each other shall be constant.

1846 A and B are playing at a game in which the chance of A's winning is $\frac{m}{m+n}$; find the chance that A will win all B's money, supposing A and B to commence with £a and £β respectively, and the stake to be £1 each game; the play being concluded when either party has lost all.

CALCULUS OF FUNCTIONS.

1839 Solve the functional equation $\phi(x) + \phi\{x + \phi(x)\} = 0$.

1842 If $\log^y\{f(x)\} = f(x - y)$, and $f(0) = e$, determine $f(x)$.

Shew that $\frac{\theta}{\theta'} = 2.\left(\frac{\sin\theta}{\sin\theta'} - \frac{\sin 2\theta}{\sin 2\theta'} + \frac{\sin 3\theta}{\sin 3\theta'} - \ldots\right)$.

1846 1. If $A_1\sin x + A_2\sin 2x \ldots + A_n\sin nx + \ldots$ is the expansion of an arbitrary function of x, $f(x)$ between the limits $x = 0$ and $x = \pi$, and if $f(x)$ is continuous, and $f'(x)$ not infinite betweeen those limits, prove that

$$f(0) = \frac{\pi}{4}(L + L'), \quad f(\pi) = \frac{\pi}{4}(L - L'),$$

L or L', according as n is odd or even, being the limit of nA_n as n increases beyond all limit.

2. A system of curves possesses the property that the subtangents are equal at points whose abscissæ are equal and of opposite signs; shew that the rectangle contained by the corresponding ordinates is constant.

Find the functional equation to a class of curves which possesses the property in question.

3. Shew that $\int_0^\infty \left(\frac{e^{-\alpha x} - e^{-\beta x}}{x}\right)^2 dx = \log \frac{(2\alpha)^{2\alpha}\,(2\beta)^{2\beta}}{(\alpha+\beta)^{2(\alpha+\beta)}}$; and prove that the functional equation $\phi(x) + \phi\left(\frac{1}{1-x}\right) = c$ does not admit of any solution except $\phi(x) = \frac{c}{2}$.

CALCULUS OF VARIATIONS.

THE shortest distance between two points on a surface never coincides with a line of curvature, unless it be a plane curve. 1839

If u is any function of $y_1\, y_2 \ldots y_n, \frac{dy_1}{dx}\frac{dy_2}{dx}\ldots\frac{dy_n}{dx}$, and is homogeneous, (and not of the first dimension,) in $\frac{dy_1}{dx}\frac{dy_2}{dx}$, &c. and if $s u dx$ is a maximum or minimum, then u = a constant. 1844

A and B are two given points in the generating line of a surface of revolution whose axis is vertical: supposing a body, acted on by gravity, to descend along the surface from A to B, find its form when the whole pressure upon it between the two given points is the least possible. 1848

Find also the form of the surface when the length of the generating line between the points A and B is also given, and point out the difference between the two results.

STATICS.

1. $ABGC$, DEF are two horizontal levers without weight, B, F their fulcrums, the end D of one lever rests upon the end 1838

C of the other, HK is a rod without weight suspended by two equal parallel strings from the points E, G. Prove that a weight P at A will balance a weight W placed anywhere on the rod HK, provided

$$\frac{EF}{DF} = \frac{BG}{BC} \text{ and } \frac{P}{W} = \frac{BG}{AB}.$$

2. P, Q are two pegs, G the center of gravity of a slender rod passing over the nearer peg P and under the farther peg Q, and just kept from sliding in the direction of its length by the friction between it and the pegs. Find the ratio of PG to PQ in terms of μ, the ratio of friction to pressure between the rod and either peg, and θ, the angle the rod makes with a vertical.

3. Three equal and heavy rods in the position of the three edges of an inverted triangular pyramid are in equilibrium under the following circumstances. Their upper extremities are connected by strings of equal lengths, and their lower extremities are attached to a hinge about which the rods may move freely in all directions. Shew that the increase of tension of the strings, corresponding with a given increase of their lengths, varies inversely as $\sin^3\theta$, where θ is the inclination of each of the rods to the horizontal plane.

4. One end of a heavy chain is attached to a fixed point (A), and the other to a weight which is placed on a rough horizontal plane passing through (A), and the chain hangs through a slit in the horizontal plane. Shew that if (l) be the length of the chain, (d) the greatest distance of the weight from (A) at which equilibrium is possible, (μ) the co-efficient of friction, and (n) twice the ratio between the given weight and that of the chain

$$e^{\frac{d}{l}} = \left\{\frac{1 + \sqrt{1 + \mu^2(1+n)^2}}{\mu(1+n)}\right\}^{\mu(1+n)}.$$

5. A paraboloid is placed with its vertex downwards and its axis vertical, between two planes equally inclined to the horizon; find the greatest ratio which the height of the paraboloid may have to its latus rectum, so that, if the solid be divided by a plane through its axis and the line of intersection of the inclined planes, the two parts may remain in equilibrium.

6. A solid generated by the revolution of a semi-circle round its diameter through an angle of 60^0 is placed upon a smooth horizontal plane; determine the plane and moment of the couple which will keep the axis of revolution of the solid in a vertical position.

1. The locus of the centre of gravity of all equal segments cut off from a parabola is an equal parabola. 1839

2. A perfectly flexible chain of variable thickness hangs in the form of a curve whose equation, taken from the lowest point is $\frac{y}{b} = \log \sec \frac{x}{b}$: find the law of variation of the thickness; and shew that the chain, hanging in the form described, offers an equal resistance to rupture at each point.

3. Find the equation of equilibrium of an elastic string acted on at every point x, y by forces X, Y; and shew that if T be the tension at the point x, y, and $d_x Y = d_y X$, $\int_r T$ is less between any two points of the string than if it had assumed any other form.

4. A light thread, whose length is $7a$, has its extremities fastened to those of an uniform heavy rod, whose length is $5a$; and when the thread is passed over a thin round peg, it is found that the rod will hang at rest provided the point of support be anywhere within a space (a) in the middle of the thread: determine the co-efficient of friction between the thread and the peg; and when the rod hangs in a position bordering upon motion, find its inclination to the horizon and the tensions of the different parts of the string.

5. A thin inextensible cord, in which the density of the material increases in geometric, as the distance from one extremity increases in arithmetic ratio, is laid directly across a rough horizontal cylinder, the semi-circumference of whose base is equal to the length of the cord: find what the co-efficient of friction must be, that the cord may just be supported with its two extremities in one horizontal plane; and when the equilibrium is destroyed by any cause, find the accelerating force.

6. In finding the position of equilibrium of an uniform elliptical lamina, whose centre is C, placed between two inclined planes intersecting in (A) at right angles, we have $AC = \sqrt{a^2 + b^2}$, and the height of (C) above (A) will be a maximum when AC is vertical, which position therefore of AC ought to give the position of equilibrium. Explain the fallacy of this reasoning, and from the true solution of the problem shew that the axes of the ellipse will be respectively parallel to the planes.

7. Three rigidly connected bodies m, m', m'' are respectively attached to three equal weights hanging freely at the extremities of three strings which pass over three fixed pegs, the directions of the strings forming the three edges of a frustrum of a regular tetrahedron of which $m\ m'\ m''$ is the horizontal base. Shew that $m' + m'' = 2m$, the system $m\ m'\ m''$ will begin to move round a line in the plane $m\ m'\ m''$ perpendicular to $m'\ m''$.

1840 1. An inextensible string binds tightly together two smooth cylinders whose radii are given; find the length of the string, and the ratio of the mutual pressure between the cylinders to the tension by which it is produced.

2. A sphere rests upon a string fastened at its extremities to two fixed points: shew that if the arc of contact of the string and sphere be not less than $2 \tan^{-1} \frac{4}{5}\frac{8}{5}$, the sphere may be divided into two equal portions by means of a vertical plane without disturbing the equilibrium.

3. A uniform rough cylinder is supported, with its axis horizontal, equally upon each of the two elastic strings whose weights are inconsiderable; the strings are equal in all respects, and attached to points of the same horizontal plane above the cylinder, so that they hang vertically, and each lies entirely in a plane perpendicular to the axis of the cylinder. Find how much the cylinder descends by the stretching of the strings.

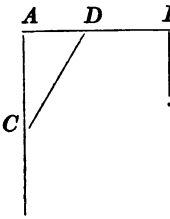
4. Two weights P and Q are attached to a third weight W by means of two strings PAW, QAW, passing through a small ring A, below which W hangs freely; determine the nature of the curve on which P and Q will rest in all positions

compatible with the conditions of the system, and deduce the particular case in which all the weights P, Q and W are equal.

5. When two pith balls repelling one another with forces varying as $\frac{1}{(\text{dist.})^2}$ are connected by a fine thread passing over a fixed point A, and are also acted on by gravity, all the possible positions of equilibrium of P and Q will pass through two ellipses having a common focus A and major axis vertical.

6. In plane Trigonometry, Prob. 6. Determine the forces to be counteracted, in order that each of the outside shot in the lowest tier may be kept in its place.

1. A and C are fixed points, or pegs, in the same vertical line, about which beams AB, CD are freely moveable; AB is supported in a horizontal position by CD, and has a weight suspended at B; find the pressure at C, (1) when there is a hinge-joint at D, (2) when CD forms one piece with AB, the weights of the beam being in each case neglected.



1841

✓2. A straight uniform beam is placed upon two rough planes whose inclinations to the horizon are a and a', and the co-efficients of friction $\tan\lambda$ and $\tan\lambda'$; shew that if θ be the limiting value of the angle of inclination of the beam to the horizon at which it will rest, W its weight, and R, R' the pressures upon the planes,

$$2\tan\theta = \cot(a' + \lambda') - \cot(a - \lambda),$$

and $$\frac{R}{\cos\lambda \sin(a' + \lambda')} = \frac{R'}{\cos\lambda' \sin(a - \lambda)} = \frac{W}{\sin(a - \lambda + a' + \lambda')}.$$

3. A uniform and straight plank rests with its middle point upon a rough horizontal cylinder which is fixed, their directions being perpendicular to each other. Find the greatest weight that can be put upon one end of the plank without its sliding off the cylinder.

4. A cone rests with its base upon the vertex of a given paraboloid, find the greatest ratio which the height of the cone

can bear to the length of the latus rectum of the paraboloid, while the equilibrium remains stable.

1842 1. A quadrilateral formed of four unequal beams jointed together at their extremities is compressed by a given force in the direction of one diagonal: find the force in the direction of the other diagonal which will resist the compression.

2. An elastic string passes over a tack, and is attached to the extremities of an uniform beam; supposing the increase of length of the string when stretched by the weight of the beam equal to twice the length of the beam, determine the position of equilibrium.

1843 1. One end of a beam, whose weight is W, is placed on a smooth horizontal plane; the other end, to which a string is fastened, rests against another smooth plane, inclined at an angle (α) to the horizon; the string passing over a pulley at the top of the inclined plane hangs vertically, supporting a weight (P). Shew that the beam will rest in all positions if a certain relation hold between PW and α.

2. A cylinder rests with its base on a smooth inclined plane, a string attached to its highest point, passing over a pulley at the top of the inclined plane, hangs vertically and supports a weight; the portion of the string between the cylinder and the pulley is horizontal: determine completely the conditions of equilibrium.

3. A heavy uniform rod, suspended at its highest point, is capable of oscillating freely in a vertical plane; each particle of it is attracted by a force emanating from a centre situated at a height (a) directly above the point of suspension, a being also the length of the rod. If μr be the law of the force, shew that the equilibrium is stable or unstable according as μa is less or greater than g.

4. Two heavy triangular laminæ have a common horizontal base, about which they are capable of revolving freely; and their vertices are joined by an elastic string. Find the position

of unstable equilibrium in which they will rest; and the increase of length of the string.

5. A heavy elastic string is laid upon a smooth double inclined plane in such a manner as to remain at rest; find how much the string is stretched.

1. A cylinder rests with the centre of its base in contact 1844
with the highest point of a fixed sphere, and four times the altitude of the cylinder is equal to a great circle of the sphere; supposing the surfaces in contact to be rough enough to prevent sliding in all cases, shew that the cylinder may be made to rock through an angle 90°, but not more, without falling off the sphere. The base of the cylinder is supposed to be sufficiently large.

2. What is meant by neutral equilibrium? Shew that the equilibrium of the cylinder, in the preceding problem, is both neutral and unstable when its altitude is equal to the diameter of the sphere. Give an example of neutral equilibrium which is also stable.

3. A cylinder, with its base resting against a smooth vertical plane, is held up by a string fastened to it at a point of its curved surface whose distance from the vertical plane is h. Shew that h must be $> b - 2a \tan\theta$, and $< b$, where $2b$ is the altitude of the cylinder, a the radius of the base, and θ the angle which the string makes with the vertical.

cc
1856

4. In Dynamics of a Rigid Body, (Problem I.) shew that, if ω be variable, the cylinder will upset, when the ratio of its diameter to its height is less than

$$\frac{c}{g}\sqrt{\omega^4 + \left(\frac{d\omega}{dt}\right)^2}.$$

5. A mass M of elastic fluid is confined in a slightly elastic spherical envelope whose natural radius is R. The tension being as the linear extension, shew that the increment of the radius is approximately $\frac{3\mu k}{8\pi}\frac{M}{R}$, μ and k being constants depending on the nature of the envelope and of the fluid.

6. A system consists of n equal particles which have no initial velocities. Prove that it will remain at rest, if their co-ordinates can only vary subject to the condition

$$n\Sigma(x^2+y^2+z^2)-(\Sigma x)^2-(\Sigma y)^2-(\Sigma z)^2 = \text{a constant}:$$

the particles attracting one another with a force which varies as the distance.

7. A flat board in the form of a square is supported upon two props with its plane vertical: determine its positions of equilibrium; friction being neglected, and the distance between the props being equal to half a side of the square.

8. One end of a beam can turn in every direction about a fixed point. The other rests on the upper surface of a rough plane, (coefficient of friction μ) which is inclined to the horizon at an angle α. If β be the angle the beam makes with the plane, prove that the beam will rest in any position if $\tan\alpha$ be not greater than $\dfrac{\mu}{\{1+\mu^2\tan^2\beta\}^{\frac{1}{2}}}$.

9. Hydrostatics, Prob. 3.

1845 1. A uniform sphere moveable about a fixed point in its surface rests against an inclined plane; find the pressure on the fixed point.

2. A pack of cards is laid on a table; each projects in the direction of the length of the pack beyond the one below it; if each projects as far as possible, prove that the distances between the extremities of the successive cards will form an harmonic progression.

1846 1. Two equal weights P, P' are connected by a string which passes over a rough fixed horizontal cylinder; compare the forces required to raise P, according as P is pushed up or P' pulled down.

2. Two equal particles are connected by two given strings without weight, which are placed like a necklace on a smooth cone with its axis vertical and vertex upwards; find the tensions of the strings.

3. When a person tries to pull out a two-handled drawer by pulling one of its handles in a direction perpendicular to its front, find the conditions under which the drawer will stick fast, and shew that they are independent of the force employed.

4. Three links of a chain, A, B and C, are moveable freely along three rigid horizontal lines in the same vertical plane. If when A and C are pulled as far apart as possible, their horizontal distances from B are equal, shew that this will always be the case when they are held in any other position.

5. A uniform chain of length $2l$ is suspended from two points in the same horizontal plane; if $2a$ be the distance of the points of support, t the tension at either of those points, and c the tension at the lowest point, prove that when t, regarded as a function of l, is a minimum, $ct = al$, the unit of force being the weight of a unit of length of the chain.

6. Three forces act in equilibrium at the angles of a triangle, one bisecting the angle at which it acts, and the other two making equal angles with the side opposite to that angle: shew that the forces are as the sides opposite to their points of application, and that they will balance if turned through any equal angles in the same direction.

7. A quadrilateral is formed by four rigid rods jointed at the ends; shew that two of its sides must be parallel, in order that it may preserve its form, when the middle points of either pair of opposite sides are joined together by a string in a state of tension.

8. Assuming friction to consist of the sum of two parts, the one proportional to the pressure, and the other to the surface in contact, shew that when a rectangular parallelepiped, the edges of which are a, b and c, is supported on a given inclined plane by the least possible force acting in a given direction, we shall have $\frac{q-r}{a} + \frac{r-p}{b} + \frac{p-q}{c} = 0$; p, q, r being the values of the force in question, when the parallelepiped rests on the faces bc, ca, ab respectively.

9. A gate moves freely about a vertical axis along which it also slides, while a point in the plane of the gate and rigidly connected with it rests on a given rough inclined plane; find the limiting position of equilibrium.

1847 A circular ring without weight, and having a heavy particle attached to a point in its circumference, rolls without sliding on the concave side of a semicircle the plane of which is vertical, and the radius equal to the diameter of the ring. Shew that every position of the ring, consistent with the condition that the particle is initially somewhere in the horizontal diameter of the semicircle, is a position of statical equilibrium.

1848 1. A uniform slender rod passes over the fixed point A and under the fixed point B, and is kept at rest by the friction at the points A and B; determine the limiting positions of equilibrium.

2. Four uniform slender rods AB, BC, CD, DA, rigidly connected, form the sides of a quadrilateral figure such that the angle A is a right angle, and the points B, C, D are equidistant from each other; when the whole is suspended at the angle A, determine the position of equilibrium.

3. A string of given length is attached to the extremities of the arms of a straight lever without weight, and passes round a small pulley which supports a weight; find the position of equilibrium in which the lever is inclined to the vertical, and prove that the equilibrium is unstable.

4. Two equal strings of length l are attached to the fixed points A, B and C, D respectively, which if joined would form a horizontal rectangle; a sphere whose diameter $= AB$ is laid symmetrically upon the strings; find the position of equilibrium and the tension of either string, supposing $l > AC + \frac{\pi}{2} AB$. Shew also how the problem is to be solved when this condition is not fulfilled.

5. A uniform bent lever whose arms are at right angles to each other is capable of being enclosed in the interior

of a smooth spherical surface, determine the position of equilibrium.

1. * Two forces F and F' acting in the diagonals of a paral- 1849
lelogram, keep it at rest in such a position that one of its edges is horizontal, shew that $F \sec a = F' \sec a' = W \operatorname{cosec} (a + a')$, where W is the weight of the parallelogram, a and a' the angles between its diagonals and the horizontal side.

2. * A cubical box is half filled with water and placed upon a rough rectangular board, so as to have the edges of its base parallel to those of the rectangle; if the board be slowly inclined to the horizon, determine whether the box will slide down or topple over.

3. Two unequal weights connected by a straight rod without weight, are suspended by a string fastened at the extremities of the rod, and passing over a fixed point, determine the positions of equilibrium.

4. A smooth body in the form of a sphere is divided into hemispheres and placed with the plane of division vertical upon a smooth horizontal plane; a string loaded at its extremities with two equal weights hangs upon the sphere, passing over its highest point and cutting the plane of division at right angles; find the least weight which will preserve the equilibrium.

Determine whether the equilibrium is stable or unstable.

5. A slightly elastic string attached to two points in the circumference of the base of a right cone at opposite extremities of a diameter, is just long enough to reach over the vertex without stretching. The cone is suspended by it from its middle point, find approximately the increase of its length.

6. An equilateral triangle without weight has three unequal particles placed at its angular points: the system is suspended from a fixed point by three equal strings at right angles to each other, fastened to the corners of the triangle: find the inclination of the plane of the triangle to the horizon.

7. A piece of string is fastened at its extremities to two fixed points; determine from mechanical considerations the form

which must be assumed by the string in order that the surface generated by its revolution about the line joining the fixed points may be the greatest possible.

8. It is required to support a smooth heavy body in the form of an ellipsoid in such a manner that a given radius in the body shall be vertical, by means of support at three points: shew that if l, m, n be the direction cosines of the radius, and the equation of the ellipsoid with the cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

then the three points in question must be on the curve of intersection of the ellipsoid with the cone

$$lyz\left(\frac{1}{b^2} - \frac{1}{c^2}\right) + mzx\left(\frac{1}{c^2} - \frac{1}{a^2}\right) + nxy\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 0.$$

ELEMENTARY DYNAMICS.

1839 1. An imperfectly elastic body, acted on by gravity, is projected in any direction with a velocity (V) against a vertical plane, making an angle (θ) with the vertical plane in which the body moves: find the velocity (V') after impact; and shew that, if the ball be perfectly elastic, (α) the elevation which makes (V') a minimum, and (a) the distance of the point of projection from the given plane, $\tan\alpha = \frac{V^2 . \sin\theta}{ga}$, and $V' = V . \cot\alpha$.

2. The time of describing any portion PQ of the parabolic path of a body acted on by gravity, is proportional to the difference of the tangents of the angles which the tangents at P and Q make with the horizon.

3. From a point P on the ground equidistant between two vertical planes A and B, an imperfectly elastic ball is projected with a velocity $= \sqrt{2gh}$ towards A and reflected by it to B; find c the altitude of the highest point of B the ball can reach: and shew 1st that if α be the elevation of the direction of projection which enables the ball to attain that altitude, $\tan^2\alpha = \frac{h-c}{h}$;

2nd, that if α', α'' be two elevations such that $\tan\alpha' + \tan\alpha'' = 2\tan\alpha$, two balls projected in those directions towards A will hit the same point of B;

3rd, that if $\sin 2\alpha''' > \cot\alpha$, a ball projected in the direction α''' will hit some point of B, otherwise not; and that the chance of hitting B by the ball projected at a random elevation towards $A = \frac{2}{\pi}\sin^{-1}\sqrt{\frac{c}{h}}$.

Two billiard balls are lying in contact on the table; in what direction must one of them be struck by a third so as to go off in a given direction? The balls are supposed equal in all respects, smooth, and of the same elasticity. 1841

1. The sights of a gun are set so that the ball may strike a given object; shew that when the sights are directed to any other object in the same vertical line, the ball will also strike it. 1842

2. Two perfectly elastic balls are dropped from two points not in the same vertical line, and strike against a perfectly elastic horizontal plane: shew that their centre of gravity will never re-ascend to its original height, unless the initial heights of the balls be in the ratio of two square numbers.

3. A body is projected vertically upwards from a point A with a given velocity; find the direction in which another body must be projected with a given velocity from a point B in the same horizontal line with A so as to strike the first body.

4. The velocities at the extremities of any chord of the parabola described by a body projected obliquely and acted on by gravity, when resolved in a direction perpendicular to the chord, are equal.

5. A thin book lies on a desk: find the greatest angular velocity round a vertical axis which can be given to the desk without throwing off the book.

6. Two bevilled wheels roll together: having given the angular velocity of the first wheel and the inclination of the axes of the cones, find their vertical angles that the second wheel may revolve with a given angular velocity.

1845 A ball thrown from any point in one of the walls of a rectangular room after striking the three others returns to the point of projection before it falls to the ground. Shew that the space due to the velocity of projection is greater than the diagonal of the floor.

1847 1. If three heavy particles be projected simultaneously from the same point in any directions and with any velocities, prove that the plane passing through them will always remain parallel to itself.

2. An imperfectly elastic ball is projected with a given velocity from a given point in a smooth inclined plane; find the direction of projection in order that the ball may cease to hop just as it returns to the point of projection.

3. If v, v', v be the velocities at three points P, Q, R of the path of a projectile, where the inclinations to the horizon are α, $\alpha - \beta$, $\alpha - 2\beta$, and if t, t' be the times of describing PQ, QR respectively, shew that

$$v't = vt', \text{ and } \frac{1}{v} + \frac{1}{v} = \frac{2\cos\beta}{v'}.$$

1848 Two bodies A, B when acted on by gravity are projected, from two given points in the same vertical line, with the same velocity, and in parallel directions; shew that if A be higher than B a pair of tangents drawn to B's path from any point of A's path will intercept arcs described by B in equal times.

1849 1. *A body is projected from a given point in a horizontal direction with a given velocity, and moves upon an inclined plane passing through the point. If the inclination of the plane vary, find the locus of the directrix of the parabola which the body describes.

2. *An imperfectly elastic ball lies on a billiard table, determine the direction in which an equal ball must strike it in order that they may impinge upon a side of the table at equal given angles.

3. *A bead running upon a fine thread the extremities of

which are fixed describes an ellipse in a plane passing through the extremities, under the action of no external force; prove that the tension of the thread for any given position of the bead is inversely proportional to the square of the conjugate diameter.

4. * The centres of two equal spheres (elasticity e, radius r,) move in opposite directions in a circle (radius R) about a centre of force varying inversely as the square of the distance; determine the motion of the spheres after they have impinged, supposing that $e = \frac{r^2}{R^2 - r^2}$; and prove that the latus rectum of the conic section described after the second impact will be $2e^2 R$.

NEWTON, 1, 2, 3.

1. WHAT portion of the surface of the Earth considered spherical can a person see, who is raised to such a height as to loose $\frac{1}{n}$th part of his weight? 1838

2. Integral Calculus, Prob. 2.

1. If $l_1, l_2, l_3, l_4, l_5, l_7$ be the Saturnocentric longitudes of six satellites of Saturn, whose periodic times are ·943; 1.37; 1.888; 2.739; 15.945; 79.33 days respectively; shew that $l_1 + 3l_2 - 2l_3 - 6l_4 + l_5 - 5l_7$ varies about $1^\circ.25'$ in the course of a year, the orbits of the satellites being supposed circular. 1839

2. If v, v', v'', be the velocities at three points P, Q, R of an ellipse described by a body round a centre of force in the centre, such that the deviation α of the direction of the motion from P to Q is equal to the deviation from Q to R; shew that if t, t' be the times of describing PQ, QR, respectively,

$$v'' \sin(t\sqrt{\mu}) = v \sin(t'\sqrt{\mu}); \text{ and } \frac{\cos(t\sqrt{\mu})}{v} + \frac{\cos(t'\sqrt{\mu})}{v''} = \frac{2\cos\alpha}{v'}.$$

If PQ, QS be arcs described in equal times, the velocities at P and S resolved perpendicular to the tangent at Q are equal.

Infer the corresponding results when the force is repulsive,

and when the force is constant and acts in parallel lines, and prove their correctness.

3. If the Earth's orbit be taken an exact circle, and a Comet be supposed to describe round the Sun a parabolic orbit in the plane of the ecliptic: shew that this Comet cannot possibly continue within the Earth's orbit longer than the $\left(\frac{3\pi}{2}\right)^{\text{th}}$ part of a year.

1840 If ρ, ρ' be the radii of curvature at the extremities of two conjugate diameters of an ellipse, $\rho^{\frac{2}{3}} + \rho'^{\frac{2}{3}}$ is constant. Also if c, c' be the *curvatures* at two points at which the tangents are at right angles $c^{\frac{2}{3}} + c'^{\frac{2}{3}}$ is constant.

1845 Two particles move in different planes about a centre which attracts with a force varying inversely as the square of the distance, the one in a circle, the other in an ellipse; the orbits have two points in common, and at either of these points the velocity of one particle is to that of the other as n to 1. Determine the eccentricity of the ellipse.

1848 If a body be projected with a given velocity about a centre of force which $\propto \frac{1}{(\text{dist.})^2}$, shew that the axis minor of the orbit described will vary as the perpendicular from the centre of force upon the direction of projection; and determine the locus of the centre of the orbit described.

1849 1. *The circle described through two points of an equiangular spiral and the point of intersection of the tangents at those points will pass through the pole. Prove this, and apply the proposition to shew that the curvature at any point of an equiangular spiral varies inversely as the distance of the point from the pole.

2. A particle describes an ellipse about a centre of force in the focus S; about S as centre a circle is described, which is cut by the radius vector SP in the point Q; from Q a line is drawn perpendicular to the direction of the particle's motion, which meets the major axis in R; prove that R is constant in

position, and that QR is proportional to the particle's velocity throughout the motion.

3. If P be a point in a cycloid and O the corresponding position of the centre of the generating circle, shew that PO touches another cycloid of half the dimensions.

DYNAMICS. PARTICLE.

1. A HEAVY particle is attached by means of a string to a 1838
point (A) in a given inclined plane which is perfectly smooth. Supposing the string to be just capable of sustaining the weight of the particle, and the initial position of the string to be horizontal, determine the motion on the plane; and find the distance from (A), measured along the inclined plane, at which the particle will pass through a vertical plane drawn through (A).

2. Two small equal weights are fastened to a string at equal distances from its ends, which are fastened to two points in a horizontal plane, the distance between which is greater than the distance between the weights; find the time of a small oscillation of the weights, in a vertical plane through the ends of the string.

1. If a heavy point be projected horizontally with a given 1839
velocity, in a medium of which the resistance is proportional to the product of its own density and the square of the point's velocity; find how the density of the medium must vary that the path may be a circle, and determine the whole motion.

2. A circular horizontal lamina of matter, every particle of which attracts with a force varying as $\frac{1}{\text{dist.}}$, is made to revolve with an uniform angular velocity (ω) round an axis through its centre perpendicular to its plane; shew that a particle of matter will move freely along a groove (in the circle) whose equation is

$$r = a \cos \sqrt{\frac{f}{\omega^2 a}}\, \theta.$$

Also, supposing the force repulsive and a string coiled round

the circle now supposed fixed, to the extremity of which is attached a particle P, shew that when the string has been unfolded from an arc subtending an angle θ at the centre, (P's vel.)$^2 = fa \log_e(1 + \theta^2)$ where f is the force at the circumference of the circle, and a its radius; find also the tension of the string in any position of P.

1840 1. The highest point of the wheel of a carriage, rolling on a horizontal road, moves twice as fast as each of two points in the rim, whose distance from the ground is half the radius of the wheel.

2. Find the rate at which the carriage is travelling, when the dirt thrown from the rim of the wheel to the greatest height, reaches a given level. Explain the two roots given by the resulting equation. If the velocity of the carriage be less than that due to a height equal to half the radius of the wheel, what is the greatest height to which the dirt is thrown?

3. Determine the motion of two weights connected by an elastic string, and moving along the convex surface of a cycloid having its axis vertical.

4. Supposing the Earth a homogeneous spheriod of equilibrium, the time of descent, of a body let fall from any point P on the surface down a hole bored to the center C, varies as CP, and the velocity at the center is constant.

5. Two imperfectly elastic spheres attract one another with forces varying as $\frac{1}{D^2}$; find the greatest separation of their centres after n impacts, their original distance being given.

6. A body is placed within a hollow cube, and attracted to its angular points by given forces varying as the distance; determine the motion.

7. Newton, 1, 2, 3. Prob. 1.

1841 1. If two bodies be projected from the same point, at the same instant, with velocities v and v_1, and in directions α and α_1, shew that the time which elapses between their transits through

the other point which is common to both their paths is equal to

$$\frac{2}{g} \cdot \frac{v v_1 \sin(a \smile a_1)}{v_1 \cos a_1 + v \cos a}.$$

2. A plane revolves uniformly about a vertical axis through which it passes, and a body descends with its flat smooth surface pressing against the plane; find the equation to the path of the body upon the plane.

3. A weight is attached to a fixed point on a smooth horizontal plane by an elastic string, and is made to revolve on the plane about the fixed point; supposing the initial position of the string to be a straight line, and the original velocity to be impressed in a direction perpendicular to that of the string, find the path of the weight.

4. The moment of inertia of any plane figure about any axis equally inclined to the principal axes, which have the same origin, is equal to two thirds of the greatest of the moments about those principal axes.

1. Two balls, attracting one another with forces varying as 1842
the distance, are constrained to move in two grooves at right angles to one another, and start from rest at equal distances from the intersection of the grooves: at the expiration of a given time they are suffered to move freely: find the equation to the path described.

2. A particle is constrained to move in a straight line, and is attached to an elastic string in such a way that the string is unstretched when perpendicular to the given line: shew that the time of a small oscillation may be expressed by an elliptic function of which the modulus is $\sin 45°$.

1. In a curve described by a body in one plane by the 1843
action of a central force, the angle between the radius vector and the tangent varies as the time. Find the curve and law of force.

2. A particle attached to one end of an elastic string moves on a smooth horizontal plane, the other end of the string being

fixed to a point in that plane. If the path of the particle be a circle, shew that $\tau^2 \propto \frac{ra}{r-a}$, a being the natural, r the stretched length of the string, and τ the periodic time.

If the orbit be nearly circular, determine the angle between the apsides.

1844 Find formulæ for determining how a given curve must be placed, in order that a heavy particle may descend along it, from one given point of the curve to another, in the shortest possible time: (1) in the case of a plane curve, whose plane is vertical; (2) in that of a curve of double curvature.

1845 1. A body P moves on the surface of a right cone, the axis of which is vertical and meets a horizontal plane in S. Find the law of force to S by which a body may be retained in the curve which is the projection of P's path on the plane.

2. In a medium, the resistance of which is partly constant and partly varying as an integral power of the velocity, find the velocity of a projectile at any point of its path in terms of the inclination of the path to the horizon at that point.

3. A comet in moving from one given point to another throws off at every instant small portions of its mass, which always bear the same ratio (n) to the masses which remains. If v be the velocity with which each particle is thrown off, α the inclination of its direction to the radius vector, prove that the period (t) will be diminished by

$$\frac{3ntv}{fa}\{(\phi' - \phi)\sqrt{ap} \cdot \sin\alpha - (r' - r)\cos\alpha\},$$

ϕ and ϕ' being the eccentric anomalies, r and r' the focal distances at the given points, a the mean distance, $2p$ the latus rectum, and f the force at distance a.

4. A particle is placed on the surface of an ellipsoid, in the centre of which is resident an attractive force, determine the direction in which the particle will begin to move.

5. Eight centres of force, resident in the corners of a cube, attract, according to the same law and with the same absolute

intensity, a particle placed very near the centre of the cube: shew that their resultant action passes through the centre of the cube, unless the law of force be that of the inverse square of the distance.

6. A number of equal particles, attracting each other directly as the distance, are constrained to move in parallel tubes; if the positions of the particles be given at the commencement of the motion, determine the subsequent motion of each; and shew that the particles will oscillate symmetrically with respect to the plane perpendicular to the tubes, which passed through their centre of gravity at the commencement of the motion.

1. A body describes a circular orbit about a centre of force situated in the centre of the circle; prove that the form of the orbit will be stable or unstable according as the value of $\frac{d \log P}{d \log u}$ for $u = a$ is less or not less than 3, P being the central force, u the reciprocal of the radius vector and $\frac{1}{a}$ the radius of the circle. 1846

2. A material particle placed at a centre of attraction varying as the distance, is urged from rest by a constant force, which acts for one sixth of the time of a complete oscillation about the centre, ceases for the same period, and then acts as before. Shew that the particle will then be retained at rest, and that the spaces moved through in the two periods are equal.

3. The inclination of a smooth inclined plane is such, that the pressure on it of a body, supported by a string parallel to the plane, is increased by one half when the string is cut and the body supported by moving the plane horizontally; shew that if the plane be stopped suddenly at any time after the commencement of the motion, the body will strike it after an equal interval.

1. A particle is constrained to move in a circle, and is acted on by a force tending to a fixed point and varying inversely as the distance; prove that the sum of the squares of the 1847

velocities of the particle at the extremities of any chord drawn through the centre of force is constant.

2. An elastic string, not acted on by gravity, is made to whirl with a given angular velocity round one end which is fixed; find the length to which the string will be stretched, and explain why the length becomes infinite for a finite angular velocity.

3. A particle is describing an ellipse about a centre of force in the focus; when the particle is at a given point, the absolute force is slightly diminished, find the consequent alterations of the semi-axes major, eccentricity, and position of the apse

If at the same time the square of the velocity had been diminished in the same proportion as the absolute force, shew that the position and dimensions of the orbit would have been unaffected.

4. A heavy particle is attached to a light string which passes over a horizontal cylinder; supposing the particle to perform small finite oscillations in a plane perpendicular to the axis of the cylinder, find approximately the correction to the time of oscillation due to the finite arc of oscillation, and shew that it vanishes when the radius of the cylinder is to the length of string hanging down as $\sqrt{3}$ to 2.

5. A a body acted on by a central force P is moving in a medium whose resistance = c (velocity); prove that

$$\frac{d^2r}{dt^2} + P - \frac{h^2}{r^3}\epsilon^{-2ct} + c\frac{dr}{dt} = 0,$$

where h is an arbitrary constant.

6. A tube of small bore, in the form of a logarithmic spiral, revolves with a uniform angular velocity about an axis passing through its pole and perpendicular to its plane, which is horizontal, and contains a particle which moves freely in it; supposing the initial velocity of the particle relatively to the tube to be equal to the velocity of the point of the tube in contact with the particle, shew that the path of the particle is another logarithmic spiral.

1. * Two bodies acted upon by gravity are projected ob- 1848
liquely from two given points, in given directions and with given velocities; determine their position when their distance is the least possible.

2. * A railway train is going smoothly along a curve of 500 yards radius at the rate of thirty miles an hour; find at what angle a plumb-line hanging in one of the carriages will be inclined to the vertical.

3. * If a body be projected from a given point in a given direction with a given velocity, and be acted on by a force tending to S and varying as $\frac{1}{(\text{dist.})^2}$, prove that if PSp be any focal chord of the body's path the sum of the squares of the velocities at P and p will be constant.

4. * A number of balls of given elasticity A, B, C... are placed in a line; A is projected with a given velocity so as to impinge on B; B then impinges on C, and so on; find the masses of the balls B, C...in order that each of the balls A, B, C.. may be at rest after impinging on the next; and find the velocity of the n^{th} ball after its impact with the $(n-1)^{\text{th}}$.

5. * An imperfectly elastic ball is projected in a given direction within a fixed horizontal hoop, so as to go on rebounding from the surface of the hoop; find the limit to which the velocity of the ball will approach; and shew that it will attain this limit at the end of a finite time.

6. If a and na be the respective distances of a satellite and of the Sun from a planet, p and mp the periodic times of the satellite and planet, which are supposed to describe circles round the planet and the Sun respectively, shew that the orbit of the satellite will always be concave towards the Sun provided n be greater than m^2.

7. A given inelastic mass is let fall from a given height on one scale of a balance, and two inelastic masses are let fall from different heights on the other scale, so that the three impacts take place simultaneously; find the relations between the masses

and heights in order that the balance may remain permanently at rest.

53 8. A body of given elasticity is projected with a given velocity, and rebounds n times at a horizontal plane passing through the point of projection; determine the direction of projection so that the angle between the direction of projection and the direction of the ball immediately after the last impact may be the greatest possible.

53 9. A cannon ball is fired at a mark at a place whose north latitude is l; shew that in consequence of the earth's rotation the vertical plane containing the axis of the cannon must be inclined at an angle of $15\, t \sin l$ seconds to the left of the vertical plane passing through the mark, t being the time of flight expressed in seconds.

53 10. Two given masses are connected by a slightly elastic string, and projected so as to whirl round; find the time of a small oscillation in the length of the string.

Give a numerical result supposing the masses to weigh 1 lb., 2 lbs., respectively, and the natural length of the string to be 1 yard, and supposing that it stretches $\frac{1}{10}$ inch for a tension of 1 lb.

53 11. A body is acted on by a force $= \frac{\mu}{(\text{dist.})^2}$ tending to a fixed centre S: shew that in general there will be two directions, differently inclined to AS, in which the body may be projected from a given point A with a given velocity v so as to pass through another given point B.

Prove also that if t, t' be the times of moving from A to B in the two cases,

$$\text{either } t = t', \text{ or } t + t' = 2\pi\mu \left\{\frac{2\mu}{SA} - v^2\right\}^{-\frac{3}{2}}.$$

53 12. Force varying as $\frac{1}{(\text{dist.})^2}$, shew that, when the latus rectum is given, an angle $= 2 \tan^{-1}\sqrt[4]{5}$ measured from the nearer apse will be described very nearly in the same time

whether the body moves in an elliptic or an hyperbolic orbit whose eccentricities are $1 - a$ and $1 + a$ respectively, a being small.

53 1. A particle is attached to the extremity of a fine string 1849
which is partially wound round a cylinder of diameter c; if the unwound portion of the string be kept stretched, and the particle be projected perpendicularly to its length with a velocity V, prove that the string will be wound up after the lapse of the time $\frac{l^2}{Vc}$, where l is the length of string unwound at the time of projection.

53 2. A body moves from rest at a distance a towards a centre of force, the force varying inversely as the distance: shew that the time of describing the space between βa and $\beta^n a$ will be a maximum if $\beta = \frac{1}{\frac{1}{n^{2(n-1)}}}$.

53 3. If the equation for determining the apsidal distances in a central orbit contain the factor $(u - a)^p$, shew that $u = a$ cannot correspond to an apse unless p be of one of the forms $4m + 2$ or $\frac{4m+2}{2n+1}$. If the factor $u - a$ occur twice then a will be a root of the equation

$$\phi(u) - h^2u^3 = 0$$

where $\phi(u)$ is the central force.

53 4. Each particle of two indefinite straight lines, lying in the same plane, attracts with a force which varies inversely as the distance. Determine the motion of a body projected in any direction along the plane.

DYNAMICS. RIGID BODY.

1. A LOADED balance being in equilibrium, one pan is made 1838
to oscillate through a small given angle; find approximately the weight which must be put into the other pan in order to restore the equilibrium.

2. A straight and heavy rod oscillates about one extremity in a vertical plane; shew that if (ϕ) be the angle included between the direction of the rod, and the direction of the pressure upon the axis at any time, and (θ) the inclination of the rod to the horizon at the same time;

$$\tan\phi \,.\, \tan\theta = \text{constant}.$$

3. A sphere the radius of which is 2 feet, is revolving uniformly about a fixed diameter at the rate of 6 times in one second, when a constant tangential retarding force begins to act upon it in a diametral plane perpendicular to the axis of rotation: the sphere is at rest after 36 complete revolutions from the instant the retarding force began to act upon it: compare this force with the force of gravity at the Earth's surface.

1839 1. Nicholson's hydrometer whose weight is w is slightly depressed until the weight of the fluid displaced becomes $w + w'$; when the hydrometer, in rising, reaches its position of equilibrium, a small weight w'' is placed in the upper dish, determine the motion.

If an additional weight $= \sqrt{w'^2 + w''^2 + \frac{w''}{w} w'^2}$ be inserted at a proper time, the instrument may be reduced to a state of equilibrium.

2. If a given force, acting horizontally, be found sufficient to keep a cylindrical roller moving with a velocity (v) over a surface which is covered with water of a depth equal to half the radius of the cylinder: find at what inclination to the horizon the same force would have to act in order to keep the same roller moving with a velocity $\left(\frac{v}{2}\right)$ if the water rose up to the axis of the cylinder, the resistance being supposed proportional to the square of the velocity.

3. An uniform rod of length ($2a$) slides in a vertical plane between two rough planes, one of which is vertical and the other horizontal; the friction on each rough plane is equal to the pressure, and the rod starts from the vertical position with

a given velocity; determine the motion, and shew that if the given velocity be just large enough to enable the rod to reach the horizontal position, the angular velocity at the inclination (θ) to the horizon will

$$= \sqrt{\frac{6}{37}\frac{g}{a} \cdot (e^{-6\theta} + 6\sin\theta - \cos\theta)}.$$

4. In Hydrostatics, Prob. 1.

If the cylinder be depressed vertically through a given space, determine the motion.

5. If a piston, whose mass is M, be placed at a distance (a) from one end of a horizontal cylinder, and the space between that end and the piston be suddenly filled with sufficient steam to produce a pressure (P) upon the end of the piston as soon as the steam is completely enclosed: prove that the velocity of the piston at a distance (x) from its original position will $= \sqrt{\frac{2Pa}{M}\log_e\left(1+\frac{x}{a}\right)}$, if the friction be neglected; and find the velocity which would be destroyed by a very slight friction.

1. A regular tetrahedron is moveable round a fixed vertical axis passing through two rings at the extremities of one of its edges; determine the least angular motion which will prevent the tetrahedron from sliding down the axis, and the co-efficient of friction when the whole pressure upon the axis takes place at the upper ring. 1840

2. Adapt to the case of the Moon's motion, the equations for determining the rotatory motion of a body disturbed by the attraction of another distant body, by introducing other variables. Hence shew that, in order to explain the fact of the Moon's always presenting to the Earth the same hemisphere, it is not necessary to suppose the primitive velocity of the Moon's rotation exactly equal to the motion in its orbit.

1. If a beam hang vertically by means of a string attached to its upper extremity and to a fixed point, shew that there are two ways in which the beam can oscillate so that each point in 1841

it comes to the vertical position of rest at the same instant, and find the respective times of small oscillations of this nature.

2. Two particles P and P', whose masses are m and m', are connected by a straight rod, and, being constrained to move in two straight grooves AP, AP' which are inclined to the horizon at given angles, make small oscillations in a vertical plane; shew that the length of the isochronous pendulum is $\frac{m \cdot OP^2 + m' \cdot OP'^2}{(m+m')\, GH}$, where OP, OP' are perpendiculars to AP, AP', and GH is the depth of the centre of gravity of P and P' below the horizontal plane drawn through A, when the system is in equilibrium.

3. If the density of a straight rod AB vary as (dist.)^n from one end A, and k, k' be the radii of gyration of the rod round A and B respectively, shew that k^2 is always some exact multiple of k'^2, whatever be the integral value of n; find also the value of n so that k may be equal to $6k'$.

4. A circular hoop has communicated to it a velocity of translation, parallel to a given rough inclined plane with which it is in contact, and in the downward direction; and also at the same time a velocity of rotation about its centre. The motion takes place in a plane which is perpendicular to the intersection of the inclined plane with a horizontal one. Find the conditions under which the hoop will descend to a given point on the inclined plane and then just return to the place whence it set out; also point out the circumstances of the remainder of the motion.

5. In Statics, Prob. 3. supposing the weight suddenly removed, find the time of the small oscillations of the plank.

1842 1. A carriage moves on a railroad with a given velocity round a curve of given radius: find the amount by which the outer rail must be elevated above the inner one, in order that the carriage should not be overturned towards the outside.

2. An inelastic cylinder rolls down a rough inclined plane, and strikes against another plane equally inclined to the horizon

but in the opposite direction: shew that if the angle between the planes be less than 60° the sphere will, after the blow, be reduced to rest.

1. A screw of Archimedes is capable of turning freely round its axis, which is fixed in a vertical position; a heavy particle is placed at the top of the tube and runs down through it; determine the whole angular velocity communicated to the screw. 1843

2. A thin spherical shell, whose radius is a, rests on a smooth horizontal plane; a particle of the same mass as the shell, is placed at the lowest point of its internal surface, which is smooth: shew that, if the shell be projected with a horizontal velocity $2\sqrt{(ga)}$, the particle will ascend just as high as the centre of the shell, and then descend.

3. If the velocity of projection be greater than $2\sqrt{(ga)}$, determine completely the nature of the motion so far as velocities are concerned.

4. A and C moments of inertia of the terrestrial spheroid about its equatorial and polar diameters respectively; shew that

$$\frac{C-A}{A} = \frac{\int_0^{a_1} \rho\, d(a^5 e)}{\int_0^{a_1} \rho\, d(a^5)}.$$

5. A prolate spheroid rests symmetrically on two rods equally inclined to the horizon in different vertical planes, their point of concourse being lowest; find the position of equilibrium, and the time of a small oscillation.

6. An ellipsoid rests with its least axis (c) vertical on a rough horizontal plane; when slightly disturbed in any manner, find the path of the centre of gravity. Shew that the coexistent oscillations in principal planes cannot be synchronous, but that the principal planes will attain maximum amplitudes together when

$$\frac{\text{number of oscillations in plane } (a, c)}{\text{number of oscillations in plane } (b, c)} = \left(\frac{a^2 - c^2}{b^2 - c^2} \cdot \frac{b^2 + 6c^2}{a^2 + 6c^2}\right).$$

7. AP, BQ are two arms moving round the fixed centres A and B, and the points P and Q are connected by a link

(*i. e.* a rod) PQ; shew that the angular velocities of the arms AP, BQ are inversely proportional to the segments into which the link divides the line of centres (*i. e.* AB).

8. Two wheels revolving uniformly in the same plane are suddenly brought in contact, and their axes are kept fixed; the friction being sufficient to prevent all sliding, determine what alteration is made in the angular velocities.

9. An elliptic board has the extremity of its minor axis fixed; it is struck in a direction perpendicular to its plane through one focus. If the eccentricity $= \frac{1}{\sqrt{(5)}}$, the axis of initial rotation will pass through the other focus.

1844 1. A beam is moving in a vertical position with a given horizontal velocity, and its upper extremity becomes suddenly fixed; determine the subsequent angular velocity of the beam about its fixed extremity.

2. A rough horizontal plane is constrained to revolve round a vertical axis, with a constant angular velocity ω, and a uniform cylinder is placed with its base upon it; shew that, if the friction be sufficient to prevent sliding, the cylinder will upset, when the ratio of its diameter to its height is less than $\frac{c\omega^2}{g}$, c being the distance of the axis of the cylinder from the axis of rotation.

3. Four equal particles (exercising no attraction upon each other) move in the same ellipse by the action of a central force in the centre; at the commencement of the motion they were situated at the extremities of the major and minor axes ($2a$ and $2b$): shew that, if at any time they become suddenly connected with each other, so as to form a rigid system, this system will revolve about the centre with a constant angular velocity equal to

$$\frac{2ab\sqrt{\mu}}{a^2+b^2},$$

μ being the absolute force at the centre.

4. A smooth circular ring, placed on a smooth horizontal plane, has a point A of its interior circumference fixed, and, at the other extremity of the diameter terminating at A, a particle is placed: if any angular velocity (ω) be communicated to the ring, shew that the particle will come to rest, after describing a certain arc of the ring, and that the ring will then revolve uniformly with the angular velocity ω.

5. A uniform rod moves about a fixed point in contact with one of its extremities, in such a manner that the vertical plane passing through the rod has a uniform angular motion in azimuth. Investigate the rod's small oscillations in altitude.

6. Two bodies move round fixed axes, and their moments of inertia are equal. Shew that when in any manner they exert impulsive action on each other, the angular velocity which either has when the impulsive action is complete, is proportional to that which the other has lost or gained; the system being inelastic.

7. A beam, placed upon a smooth horizontal plane, has one extremity fixed; and a ball (A) is placed in contact with it at a given distance from the fixed extremity: determine at what point another ball must impinge directly upon it, so that the greatest possible velocity may be communicated to A by the impact. The beam and balls are inelastic.

8. A rigid body, one point of which is fixed, is struck by a certain couple: if any other point (xy) of the body situated in a certain plane had also been fixed, the initial *vis viva* would have been $\frac{(mx + ny + p^2)^2}{ax^2 + by^2 + c^3}$. Determine the values of x and y for the point in which the axis of spontaneous rotation meets the plane; a, b, c, and m, n, p being given constants.

1. A circular disc revolving about an axis through its centre perpendicular to its plane, which is inclined at a given angle to the horizon, is placed upon a smooth horizontal plane; determine the motion. 1845

When the initial velocity of the circumference is very great compared to that acquired in 1″ by a falling body, find the time and extent of the vertical oscillations of the centre of gravity.

2. A smooth triangular board ABC, of which the angle C is obtuse, is fixed with its plane vertical, and the side AB resting on a horizontal plane: a weight is allowed to run down the side CA; the board is suddenly unfixed; compare the pressure of the weight on the board before and after it is allowed to move.

3. Two equal heavy balls (A and B) are thus suspended; the ball (A) by a fine thread from a fixed point, and the ball (B) from the lowest point of the ball (A) by another fine thread of the same length as the former. The ball (B) receives a slight horizontal blow, determine the motion.

4. A smooth oblate spheroid, the centre of which is fixed, is revolving uniformly about its axis of figure, when it receives a normal blow; determine the subsequent motion, and shew that the instantaneous axis of rotation will always lie in the surface of a right cone.

5. In ~~Dynamics of a rigid~~ Statics body. Prob. I.

Supposing the diameter which passes through the fixed point to be horizontal, shew that if the plane be suddenly removed the pressure will be increased or diminished according as the inclination of the plane was less or greater than $\tan^{-1} \frac{1}{2}$.

1846 1. A rigid body is attached to a fine wire, and made to perform small oscillations by the torsion of the wire; regarding for simplicity the length of the wire as infinite compared with the dimensions of the body, form the equation one of the roots of which gives the time of oscillation.

Show that the time of oscillation which would be obtained by considering only the moment of the force of torsion, and the moment of inertia of the body about a vertical axis passing through its centre of gravity, would in general be erroneous unless that axis be a principal axis. Find also the time of

oscillation to a second approximation, supposing the force of torsion very small.

2. A material particle is projected along a smooth grove in a vertical board which slides freely along a horizontal plane; find the form of the groove that the velocity of the particle may be constant, assuming the initial motion to be vertical.

3. Two equal inelastic balls, connected by a rigid rod without weight, slide down in a vertical plane between a smooth vertical and a smooth horizontal plane. If the lower ball impinge directly upon an equal ball at rest, shew that the angular velocity of the rod will be diminished in the ratio $1 : 1 + \sin^2\beta$, where β is the inclination of the rod to the horizon at the instant of impact.

4. In playing at cup and ball, explain why spinning the ball makes it more easy to catch it on the spike.

1. A tube of small bore, in the form of a circle, revolves about one of its diameters as a fixed vertical axis, and contains a smooth particle; the angular velocity with which the tube is set in motion is known, and the particle starts from its highest position with a given velocity. Determine the angular velocity of the tube, and that of the particle relatively to the tube, for any given position of the particle, and shew that there are three other positions of the particle in the tube for which the angular velocity of the tube, and one other for which that of the particle is the same as for the given position. 1847

2. A uniform flexible and inextensible heavy string AB is laid in the form of a circular arc on a smooth horizontal plane; a given impulsive force being applied at A in the direction of the tangent, find the impulsive tension at any point, and shew that the direction of the initial motion of A makes with the direction of the force an angle whose tangent $= \frac{\epsilon^{a} - \epsilon^{-a}}{\epsilon^{a} + \epsilon^{-a}}$, a being the angle which AB subtends at the centre.

3. A narrow hoop is rolled along a rough horizontal plane in such a manner as to move nearly in a vertical plane, but make

small oscillations on each side of it; find the time of oscillation, and shew that oscillations of this sort are not possible unless the velocity of the centre of the hoop be greater than that due to one third of the radius.

4. A slender rod suspended horizontally by two equal parallel strings attached to two points equidistant from its ends oscillates round a vertical line; find the time of a small oscillation.

If in the position of equilibrium the strings are inclined at equal angles to the vertical, shew that the time of oscillation is the same as it would be if the strings were parallel, of a length equal to the projection of either of them on a vertical line, and at a distance equal to a mean proportional between their distances at the points of suspension and attachment respectively.

1848 $\int$ 1. An imperfectly elastic homogeneous rough sphere is projected obliquely, without rotation, against a fixed plane; if i, i' be the angles of incidence and reflection, λ the co-efficient of elasticity for direct impact, and ρ the ratio of the tangential forces of restitution and compression, prove that

$$2\rho = 5 - 7\lambda \tan i' \cot i.$$

2. A rough sphere rolls within a hollow cylinder with its axis vertical, so as to be in contact with the curved surface and the flat bottom; find the reactions and frictions in terms of the angular velocity with which the sphere goes round, and explain the indeterminateness of the Problem.

3. In Dynamics of a rigid body, Prob. 5.

Find the time of a small oscillation when the position of equilibrium is slightly disturbed.

4. A section of the surface of a circular right cone (whose axis is horizontal and vertical angle 60°) is formed by a plane perpendicular to the slant side so as to contain the vertex: shew that when the surface so cut off makes small oscillations about the axis, the length of the isochronous simple pendulum $= \frac{21a}{16}$ (whether the elliptic base be included in the surface or not), a

being the length of the perpendicular drawn from the vertex upon the elliptic base.

1. If a uniform inextensible string in the form of any con- 1849
tinuous curve be subjected to an impulsive tension at its extremities, the tension at any point will vary directly as the velocity communicated to that point in the direction of the radius of absolute curvature and inversely as the curvature.

2. The nut of a screw rests upon a smooth horizontal plane, over a hole cut so as to allow a free passage for the screw, and the screw descends through the nut by its own weight; determine the motion.

3. The centre of a rough sphere is fixed; if another sphere be placed on the top of it and just displaced, determine the motion of both spheres.

4. A man standing in a swing is set in motion; shew that he can accelerate the motion and increase the arc of oscillation by crouching and rising in the swing; and prove that the effect will be the greatest if he crouch when the swing is at the highest point, and rise when it is at the lowest point of its arc of oscillation.

5. A circular hoop rests upon a smooth horizontal plane with a particle at its lowest point, and receives a horizontal velocity of projection V in its own plane; find the value of V in order that the particle may just rise to the height of the centre of the hoop.

Determine the motion when V is greater, and also when it is less than this value; and find the time of an oscillation of the particle in the hoop when V is small.

6. A heavy lamina in the form of an equilateral triangle, suspended as in Problem 9 by three strings, equal but not necessarily at right angles to each other, is drawn a little aside from its horizontal position (the strings being all stretched): its centre of gravity then receives a small horizontal velocity of projection in a direction perpendicular to the plane in which the displacement was made, while at the same time a velocity of

rotation is communicated to it in its own plane: Determine the motion.

7. A man hangs by a rod which swings in a vertical plane; compare the exertion required to raise him from one given point of the rod to another, 1st, when he draws himself through a given small space always when the rod is vertical, and 2ndly, if he makes the effort when the rod is at its greatest inclination to the vertical.

8. What must be the angular velocity of a horizontal cylinder in order that a heavy string of given length attached to it may be just wound up.

9. A heavy rod is suspended from a fixed point by two inextensible strings without weight, the strings and the rod forming an equilateral triangle: if either of the strings be cut, determine the initial tension of the other.

10. A semicircular board, moving in its own plane without rotation and with its curved boundary foremost, comes in contact with a smooth fixed obstacle; determine at what point the impact should take place in order that the angular velocity generated may be the greatest possible.

11. A uniform solid cylinder is revolving with a given angular velocity about its centre of gravity, which is fixed; the cylinder receives a blow of given intensity in a direction perpendicular to the plane in which the axis moves; determine the subsequent motion.

12. A solid cone is suspended by its vertex from a point in a perfectly rough wall; if the cone be slightly displaced from its equilibrium position, the surface remaining in contact with the wall, determine the time of a small oscillation.

13. An indefinitely great number of indefinitely thin cylindrical shells, just fitting one within another, are revolving with different angular velocities, but in the same sense, about their common axis; also the angular velocity of each shell is proportional to a positive power (the n^{th}) of its radius, and that of the

outermost shell is ω. Prove that if the system of shells be suddenly united into a solid cylinder, the cylinder will revolve about it axis with the angular velocity $\frac{4\omega}{n+4}$.

If the bodies composing the solar system were suddenly to become rigidly connected, explain what the nature of the subsequent motion would be.

HYDROSTATICS.

1. A CYLINDRICAL diving bell is suspended by a rope fastened 1838
to a point at a very small distance from the center of the top of the bell: find the angle between the axis of the bell and a vertical.

2. A conical vessel is divided into two parts by a plane through its axis, and the parts are prevented from separating by an inextensible string without weight, which is a diameter to the rim of the vessel and perpendicular to the dividing plane: the parts are also hinged at the vertex. Supposing the vessel placed with its vertex downwards, its axis vertical, and filled with fluid: compare the tension of the string with the weight of the fluid.

3. A solid cone of uniform density, the axis of which is equal to the radius of its base, floats with its axis vertical, and vertex downwards in two fluids which do not mix: the axis is trisected in the points in which it meets the planes of the surfaces of the two fluids: compare the densities of the fluids in order that the equilibrium of the cone may be indifferent.

1. An uniform elastic string has it upper extremity fixed, 1839
and the lower attached to the centre of the top of a heavy homogeneous cylinder which rests partly immersed, with its axis vertical, in a fluid whose density is less than that of the solid; find the position of equilibrium.

2. If a solid globe, radius (a), be placed in the centre of a globular cavity, radius (b), and the space between the surfaces

filled with an elastic fluid, every particle of which is attracted towards the centre of the globe by a force proportional to the inverse square of the distance therefrom; and if, in the state of rest, (δ) be the density at the surface of the globe, (δ') that at the boundary of the cavity: prove that $\frac{\delta}{\delta'} = e^{m} \cdot \frac{b-a}{ba}$; and shew how ($\delta$) and ($\delta'$) are to be determined absolutely, when the mass of the whole fluid is given.

1840 1. A hollow cube, filled with heavy fluid, is held with one diagonal vertical; find the centre of pressure of one of the lower faces.

2. When the centre of gravity is at a proper depth in a fluid whose density varies as the depth, an uniform equilateral triangle will rest in any position; and a right-angled triangle may be kept with either side horizontal, by a couple whose moment will be the same whichever side and whichever angle is uppermost.

3. A thin hemispherical bowl of given weight, partly filled with fluid, is placed with its axis vertical upon the highest point of a sphere: find the nature of its equlibrium as respects stability.

4. A given quantity of mercury is taken out of the tube of a wheel-barometer; find the corresponding error of the index.

1841 1. Shew that a uniform rod of different density from that of water, when left to itself in water, must practically either float at length on the surface, or lie at length at the bottom.

A uniform rod, of less specific gravity than that of water, has a string attached to one end of it and also to the bottom of a pond whose depth is less than the sum of the lengths of the rod and string; find each position of stable equilibrium which the rod can take.

2. A hollow vessel in the form of a tetrahedron is filled with a given weight w of fluid, and is placed in an inverted position on a horizontal plane; shew that the fluid will escape from under the vessel, unless the weight of the vessel be at least equal to twice w.

3. If a prismatic diving-bell of given volume be sunk to a given depth in water, find the volume of air (at its natural density) which will be required to be forced into the bell, in order that $\left(\frac{1}{m}\right)^{th}$ of its volume may be free from water.

4. A cubical vessel full of fluid revolves uniformly about a vertical axis through one edge The upper face is the lid of the vessel, and is moveable about the angular point next the axis; find the angular velocity that the weight of the lid may keep it in its place.

5. A certain substance whose volume is v and density ρ is enclosed in a thin expansible film and sunk in water. The substance undergoes a change (in proportion to the time) such, that each portion of it swells into m times its original volume; shew that the substance will begin to rise in the water after $\frac{v(\rho-1)}{n(m-1)}$ seconds, if n be the portion of the substance changed in one second, and unity represent the density of water. Find also the portion of the substance which remains unchanged at that instant.

1. A homogeneous rectangular beam turns freely round a horizontal axis passing through the middle point of one end and parallel to a terminal edge, while the other end dips into a fluid of greater specific gravity: find the position of equilibrium. 1842

2. A thin surface in the form of a paraboloid of revolution is just immersed in a fluid, first with its base, next with its vertex downwards by means of two weights W, W': shew that if h be the height of a column of fluid whose weight is equal to the atmospheric pressure, $h = a\left(\frac{W}{W'}\right)^{\frac{1}{2}}\frac{W}{W'-W}$, a being the height of the paraboloid.

1. Two gases whose volumes, temperatures, and pressures are known, are mixed under the standard temperature and pressure; the mixture is observed to contract and cool, the pressure remaining the same; the reduced temperature being 1843

known, the whole mass is raised to the standard temperature and its volume is noted. Find the contraction of the mixture at first.

2. In a vessel full of fluid, revolving uniformly about a vertical axis, a very small sphere, lighter than the fluid, is attached by a string of given length to a given point in the base, being wholly immersed. Determine the position of the sphere and the tension of the string, when the sphere is at rest relatively to the fluid.

1844 1. A cylinder floats in water with its base four inches below the surface; when an ounce weight is placed upon it, it sinks another inch; shew that its weight is four ounces.

2. A weight is supported by two pistons which are rigidly connected together, and which work in two equal vertical cylinders, containing unequal masses of the same elastic fluid. Under what conditions will the weight continue at rest, when a communication is opened between the cylinders, supposed in contact?

3. A flexible thread is placed in a tube of any form and is acted on by any forces. The diameter of the tube is equal to that of the thread, and is infinitesimal. Determine the position of equilibrium. What difference would there be, if the thread were replaced by a filament of fluid?

4. A uniform rectangular board $ABCD$ rests, with its lowest corner (A) immersed in a fluid, its highest corner (C) held up by a string, and the lowest (B) of the other two corners at the surface of the fluid. Shew that, if the side BC make an angle 45^0 with the vertical, then

$$\frac{AB}{BC} = \frac{\text{specific gravity of the board}}{\text{specific gravity of the fluid}}.$$

1845 1. A cone, the vertex of which is fixed at the bottom of a vessel containing fluid, is in equilibrium with its slant side vertical and the lowest point of its base just touching the

surface of the fluid. Compare the density of the cone with that of the fluid.

2. A hollow sphere, filled with equal quantities of two incompressible fluids which do not mix, revolves uniformly about its vertical diameter, and the fluid particles are relatively at rest. Find the angular velocity, supposing that the lighter fluid just touches the lowest point in the surface of the sphere.

1. Two heavy particles are connected by a light string, and 1847
immersed in a fluid whose specific gravity is intermediate between those of the particles, and which revolves uniformly about a fixed axis, and is not acted on by gravity; find the position of equilibrium of the particles relatively to the fluid, and shew that when the equilibrium is possible it is stable.

2. An infinite mass of homogeneous incompressible fluid acted on by no forces is at rest, and a spherical portion of the fluid is suddenly annihilated: find the instantaneous alteration of pressure at any point of the mass, and prove that the cavity will be filled up in the time $\left(\frac{6\rho}{\omega}\right)^{\frac{1}{2}} a \int_0^1 \frac{\lambda^4 d\lambda}{\sqrt{1-\lambda^6}}$ where a is the initial radius of the sphere, and ω the pressure at an infinite distance, which is supposed to remain constant.

3. A bucket partly filled with water is attached to a weight by a string which passes over a fixed pully; supposing the water to revolve with a given angular velocity as the bucket is ascending or descending, find the form of the free surface.

4. A circular disc is suspended by a fine wire attached to the centre, and immersed horizontally in a fluid; the wire being suddenly turned through a given angle, determine the motion of the disc, supposing each element of the surface acted on by a friction varying as the velocity, and shew that the successive arcs described from rest to rest are in geometric progression. Shew also that if the friction exceed a certain quantity the disc will not come to rest at all.

5. A given quantity of incompressible fluid is contained in

an elastic spherical envelope, and justs fills the space inclosed without stretching the envelope; if the particles of the fluid be acted on by a repulsive force varying inversely as the square of the distance from the centre, find the absolute force when the space originally occupied by the fluid is left a vacuum.

1848 1. * An inverted vessel formed of a substance which is heavier than water contains enough of air to make it float; prove that if it be pushed down through a certain space, it will be in a position of unstable equilibrium; and determine the space in question.

2. * A uniform piston, terminated by a plane of area A perpendicular to its side, is inserted into an orifice in a vessel containing fluid; prove that the work done in gently pushing in the piston through a small space s is ultimately equal to the work done in lifting a portion of the fluid of volume As through a height equal to the depth of the centre of gravity of the plane below the surface of the fluid.

3. * Two equal slender rods AB, AC moveable about a hinge at A and connected by a string BC rest with the angle A immersed in a given fluid; determine the tension of the string BC.

1849 1. * A body floats in a mixture of two given fluids with a volume A immersed; one half of the mixture being removed, and its place supplied by an equal quantity of the lighter fluid, the same body floats with a volume $A + B$ immersed. Determine the ratio of the quantities of fluid in the original mixture, supposing the volume of the mixture to be equal to the sum of the volumes of the component fluids.

Explain the result when the densities of the fluids are as $A + B$ to $A - B$.

2. A cylinder the radius of which is a, having its axis vertical and containing incompressible fluid, revolves about its axis with an angular velocity $= \sqrt{n\frac{g}{a}}$, n being > 1; a sphere whose radius is also a, on being put into the cylinder, is sup-

ported in such a position that it touches the free surface at its vertex: Shew that

$$\frac{\text{density of the sphere}}{\text{density of the fluid}} = \left(1 - \frac{1}{n}\right)^3.$$

3. A plane body, one of the edges of which is a straight line, is immersed in water so as to have this straight line coincident with the surface; shew how the depth of the centre of pressure may be deduced from observation of the time of a small oscillation in vacuum of the body about its rectilinear side.

4. If XYZ be the forces acting at a point (xyz) of a mass of heterogeneous fluid in equilibrium, and $Xdx + Ydy + Zdz$ be not a perfect differential, then the pressure and density will be constant throughout the curves of which the differential equations are

$$\frac{dx}{\frac{dZ}{dy} - \frac{dY}{dz}} = \frac{dy}{\frac{dX}{dz} - \frac{dZ}{dx}} = \frac{dz}{\frac{dY}{dx} - \frac{dX}{dy}}.$$

The bob of a pendulum is a hollow sphere which can be filled 1838
with a fluid or with a solid sphere of the same density as the fluid: having given the length of the equivalent simple pendulum when the cavity is filled with the solid, find its length when filled with the fluid.

CAPILLARY ATTRACTIONS, TIDES, &c.

1. The whole height of the fall at a water-mill, and the 1841
quantity of water passing through the wheel in a given time, are given; find the relation between the height from which the water falls before it strikes the buckets, and the uniform velocity of the centre of each bucket, when the effect produced is the greatest possible.

2. A sphere of less specific gravity than water is placed at a given depth in a stream running with a given uniform velocity, and then left to the action of the stream; find the motion and path of the sphere.

3. Find an expression for the time of a small oscillation of any body about a fixed horizontal axis, the body being partly immersed in a fluid; and deduce from it the times of oscillation, (1) when the body floats freely and moves about its centre of gravity, (2) when it is wholly immersed. The resistance of the fluid is not to be taken into account.

1843 1. A closed vessel filled with an elastic fluid is let fall; determine what the law of density of the fluid must be, in order that all its particles may be at rest, relatively to each other, during the motion.

2. A solid generated by the revolution of the curve $y \propto x^{\frac{n}{2}-1}$ round the axis of x, floats in a fluid with a portion (h) of the axis immersed. If the solid be depressed through $(n^{\frac{1}{n-1}} - 1)\, h$, it will on its return just emerge from the fluid.

1844 1. A mass of homogeneous incompressible fluid, moving in a straight tube of uniform bore under the action of no forces, meets a piston which, by compressing a spring, gradually reduces the fluid to rest: if p be the pressure (on a unit of area) at any point of the fluid, whose distance, from the extremity (E) not in contact with the piston, is x; shew that, at any time t,

$$p = \frac{P}{V} x,$$

V being the volume of the fluid, and P the pressure exercised by the piston, at the time t, upon the extremity (E') of the fluid in contact with it.

It has been asserted that, when an incompressible fluid mass, moving in a tube, is suddenly reduced to rest by an impulsive force, the pressure caused by the impulse is the same at every point of the fluid: prove that this is not true.

2. In the preceding problem, if the bore of the tube be variable and small, shew that

$$p = \frac{P \int_0^x X dx}{\kappa \int_0^u X dx},$$

where κ is the area of the piston, $\frac{1}{X}$ the area of the section of the tube at the distance x from (E), and u the distance of E' from E.

1. An ellipsoid floats in stable equilibrium in a fluid of 1845
double its density. Compare the time of a small oscillation about either axis of the plane of floatation with the time of a small vertical oscillation.

2. Waves are propagated along a canal of uniform depth and breadth, the motion of the particles being small. Shew that the square of the period of the waves is equal to

$$\frac{2\pi\lambda}{g}\,\frac{e^{\frac{4\pi k}{\lambda}}+1}{e^{\frac{4\pi k}{\lambda}}-1},$$

k being the depth of the canal, λ the length of the waves, and g the force of gravity.

1. A very small capillary tube which is slightly conical is 1846
inserted vertically into a liquid which moistens its surface; prove that there are in general two positions of equilibrium of the liquid, and show in what case the equilibrium is stable.

2. A vessel with an orifice at the bottom is suspended by a string and filled with a heavy fluid; find the tension of the string for a given position of the surface of the fluid, regarding the motion of the fluid at any instant as steady, and adopting the hypothesis of parallel sections.

3. A plane is moveable at right angles to a stream, the direction of which makes an angle a with the normal, and the force required to retain it at a constant velocity v in one direction is four times that required in the opposite direction. Assuming the common theory of resistances, shew that the velocity of the stream is $3v \tan a$, or $\frac{1}{3} v \tan a$, according as the forces in the two cases are in the same or in opposite directions.

1. A cylindrical vessel with its axis vertical is filled with 1848
fluid which issues from a great number of small orifices pierced

in the side; find the surface which touches all the streams of spouting fluid.

2. A closed vessel is filled with water containing in it a piece of cork which is free to move; if the vessel be suddenly moved forwards by a blow, shew that the cork will shoot forwards relatively to the water.

3. A closed vessel is filled with water which is at rest, and the vessel is then moved in any manner; apply the principle of the conservation of areas to prove that if the vessel have any motion of rotation no finite portion of the water can remain at rest relatively to the vessel.

1849 1. Steam is rushing from a boiler through a conical pipe, the diameters of the extremities of which are D and d respectively; prove that if V and v be the corresponding velocities of the steam.

$$v = V \frac{D^2}{d^2} e^{\frac{v^2 - V^2}{2k}},$$

where k is the pressure divided by the density and supposed constant. The motion may be supposed to be that of a fluid diverging from a centre, the centre being the vertex of the cone of which the pipe forms a portion.

GEOMETRICAL OPTICS.

1838 1. A PLANE surface touches a self-luminous sphere, shew that the illumination of the surface at any point varies inversely as the cube of the distance from the center of the sphere.

2. An eye is placed under and close to the surface of a clear and stagnant fluid, and a vertical straight rod of given length is placed at a given distance from the eye: determine the form of the image, and the altitude of its highest point above the surface of the fluid.

3. If a ray of light be reflected once at each of two plane surfaces in any planes, shew that

$$\cos D = \cos 2I + 2 (\sin I \sin \phi \sin \theta)^2,$$

where D is the whole deviation of the ray, I the angle between

normals to the surfaces, ϕ the angle of incidence of the first incident ray, and θ the angle between the plane of incidence of the first incident ray, and a plane perpendicular to the intersection of the reflecting surfaces.

4. Having given the picture of a cube traced upon a plane surface by means of a camera lucida, and the point in which a perpendicular from the eye-hole of the camera lucida meets the plane of the picture; find the distance of the eye-hole from the plane of the picture and the angles which the edges of the cube make with it.

5. If the angular distance between the Sun's center and a distant station be measured with a Hadley's sextant, and the index moved forwards through an angle equal to the angle between the axis of the telescope and a normal to the fixed mirror, without moving the sextant, the Sun's light will be reflected from the moveable mirror to the distant station.

6. If a telescope, the object-glass of which is covered by a disc having two small holes in it, be pointed at a star, the image of the star will appear double except when the distance of the object-lens from the eyepiece is such as to shew the star distinctly.

1. The only surface which, when exposed to a luminous 1839
sphere, will be equally illuminated at every point, is that formed by the revolution of a portion of an arc of a lemniscata round its diameter, having the node coincident with the center of the sphere. And if the Moon (supposed spherical) be illuminated by the Sun, the intensity of light at any point of its surface, will vary nearly as the cosine of the angular distance of that point from the Sun as seen from the Moon's centre.

Reconcile these results with the general appearance of the nearly uniform brightness of the Moon's disc. And state what reasons you have for supposing that the Moon's edge should appear brightest.

2. If there be a small speck upon the middle of one of the sides of an equilateral prism, and a person place his eye close to

the opposite edge in the plane drawn through the speck perpendicular to the axis of the prism: shew that he will see *two* specks, and find the apparent angular distance between them.

3. When a cylindrical china jar, standing upon the ground receives the Sun's rays obliquely, a bright curve is observed to form itself at the bottom of the jar, and it is found that the shape and dimensions of this curve are not affected by the varying elevation of the Sun: account for this latter circumstance and determine the nature of the bright curve.

4. If a piece of fine glass of given length and in the form of a frustum of a cone, be ground so that its smaller end is spherical and concave, and its larger end spherical and convex: find what must be the ratio between the radii of these two spherical surfaces, that the piece of glass may act as a telescope for distant objects; and state the advantages and disadvantages of such an instrument as compared with the common opera-glass.

1840 1. Two semi-circular self-luminous plates are placed with their diameters upon a plane, and with their planes perpendicular to the line which joins their centres A and B; find the position of that point in the line AB, where the illumination of the plane upon which the semicircles are placed is least.

2. A pencil of rays is incident parallel to the axis x of an ellipsoid whose greatest and least axes are a and c respectively; find the nature and *limits* of the two curvilinear boundaries of the portion of the plane xy through which all the rays will pass; and shew that if e, e', e'', be the eccentricities of the principal sections in the planes xy, xz, yz respectively and $\mu = \frac{1}{e}$, the boundaries will be two ellipses whose semi-axes are ae''^2, be''^2, and $\frac{cae}{b}$, be'': but if $\mu = \frac{1}{e'}$, all the rays will pass through a *portion* of the *arc* of an ellipse, whose semi-axes are ae', be'' included between the vertex and a double ordinate at a distance $\frac{cae'}{b}$ from its centre.

3. The Sun's spots when observed through a powerful telescope exhibit phenomena which occur in the following order. Immediately surrounding a dark spot is a light ring; next to this appears a darker, usually called the penumbra, which is bounded by a yet darker line; then again a light ring, brighter than the general body of the Sun into which the alternations of light and shade at length subside.

How does this appearance favour the hypothesis of actual eminences of the Sun's body exposed to view by fluctuations in a luminous atmosphere, and seem at variance with the supposition of a dark body surroundad by a cloudy as well as a luminous atmospherical stratum?

A small pencil of rays is transmitted through a prism in 1841
a plane perpendicular to its axis; find the focus of emergent rays in the primary plane, taking account of the thickness of the prism; and deduce from the result the expression for the place of the focus when a pencil is transmitted through a plate which is bounded by parallel plane surfaces.

1. Find the quantity of light received by an infinite plate 1842
placed at a given distance from a self-luminous sphere.

2. If a concave mirror revolve round an axis through its centre, slightly inclined to the axis of the mirror, find the appearance on a screen placed at the further focus, when a series of electric sparks passes in quick succession through the nearer focus in the axis of revolution.

3. If α, β be the distances of any two geometrical foci Q, q from the surface of a spherical refracting medium, and Q', q' any other two geometrical foci, shew that

$$\frac{\beta}{\alpha} \cdot \frac{1}{qq'} - \mu \frac{\alpha}{\beta} \cdot \frac{1}{QQ'} = \frac{\mu - 1}{r}.$$

4. If a transparent plate lie between two media of unequal densities, each less than that of the plate, and if a ray, passing from the denser of the external media into the plate, fall at the critical angle on the third medium, the angle of incidence at the

first surface of the plate will be the critical angle from the first on the third medium.

1843 A small pencil of parallel rays enters a reflecting sphere, and after n internal reflexions emerges through the same aperture; find the angle of incidence, and the position of the primary focal line at emergence.

1844 In the simple astronomical telescope, shew that when the apertures of the two lenses are proportional to their focal lengths, the field of view (as seen by whole pencils) is a single point. If a convex lens, of the same aperture as the eye-glass, but of any given focal length, be placed in contact with the eye-glass, determine whether the field of view is thereby increased or diminished (the telescope is always supposed to be adjusted for distant objects).

1845 1. The thickness of a plano-convex lens is one fifth of an inch, and its breadth is two inches, find the radius of its spherical surface.

2. A plane mirror, moveable about an axis in its own plane parallel to the axis of the Earth, revolves from east to west with half the Sun's apparent diurnal motion. Shew that the direction of the reflected rays of sunlight will not be sensibly altered during the day.

3. The Sun's light is refracted through a prism, the edge of which is vertical. Find the position of the refracting surfaces in order that for a given altitude of the Sun the deviation of the rays of a given colour may be a minimum.

If z be the Sun's zenith distance, ι the refracting angle, x the angle of first incidence reduced to the horizon, μ the index for the given colour, shew that the minimum deviation D is given by the equations

$$\sin\frac{D}{2} = \sin z \sin\left(x - \frac{\iota}{2}\right) : \sin x = \mu \sin\frac{\iota}{2} \cdot \sqrt{1 + \left(1 - \frac{1}{\mu^2}\right)\cot^2 z}.$$

4. A short-sighted person has a concave lens of insufficient power; shew that he may increase the power by inclining

the lens slightly towards the line joining his eye and a distant object.

5. Having the following approximate data:

for rain water $\cos^{-1}\sqrt{\frac{\mu^2-1}{15}} = 76^\circ.40'$ and $\cos^{-1}\frac{4}{\mu}\sqrt{\frac{\mu^2-1}{15}} = 46^\circ.40'$:

obliquity of ecliptic $= 23^\circ.30'$:

latitude of London $= 51^\circ.30'$:

shew that, in the latitude of London, no portion of a tertiary rainbow can be seen by an observer, whose back is turned towards the Sun, if the Sun be distant from the summer solstice by an angle greater than that determined from the equation $\sec\theta = 2\cos 11^\circ.30'$.

6. A is the fixed reflector of a sextant, B the moveable one; produce BA, the course of a ray between the reflections, to C, making $AC = AB$; and let the reflector A be fixed at C in such a position that the ray coming from B may be reflected to the same point in the opposite limb as before. If the instrument be now employed in the usual manner, what angle must be added to the reading off?

7. A ray of light is incident from the centre of an ellipsoid, the inner surface of which is polished, and the equation of which is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$; prove that the equations to the reflected ray will be

$$\frac{x_1-x}{x\left(2\frac{p^2}{a^2}-1\right)} = \frac{y_1-y}{y\left(2\frac{p^2}{b^2}-1\right)} = \frac{z_1-z}{z\left(2\frac{p^2}{c^2}-1\right)},$$

where xyz are the co-ordinates of the point of incidence, and

$$\frac{1}{p^2} = \frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}.$$

8. Assuming the result of the preceding question, prove that all rays, which after reflection pass through the line $x = y = z$, were before reflection in the surface of the cone defined by the equation

$$yz\left(\frac{1}{b^2}-\frac{1}{c^2}\right)+zx\left(\frac{1}{c^2}-\frac{1}{a^2}\right)+xy\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=0.$$

9. A fine elastic string is tied round two equal cylinders whose surfaces are in contact and axes parallel, the string not being stretched beyond its natural length; one of the cylinders is turned through two right angles, so that the axes are again parallel, find the tension of the string, supposing a weight of 1 lb. would stretch it to twice its natural length.

1846 1. When objects are viewed by means of pencils of light which are refracted through an isosceles prism, suffering an internal reflection at the base, the objects are seen quite free from chromatic dispersion, provided the prism be not very large: prove that this must happen in the general case, that is, when the refraction is not supposed to take place in a plane perpendicular to the edges of the prism.

2. If the law of refraction were assumed to be $\phi = \mu\phi'$, shew that the approximate error in finding the point where a ray, incident near and parallel to the axis of a spherical refracting surface, cuts the axis after refraction, would be $\frac{1}{6\mu}\frac{\mu+1}{\mu-1}\frac{y^2}{r}$, using the ordinary notation.

3. Two equal prisms of the same refracting substance are placed in contact; shew that a ray of given species will be just incapable of transmission through the compound prism thus formed, when the mutual inclination of their edges is $2\sin^{-1}\frac{1}{\mu\sin\alpha}$, α being the refracting angle of each prism.

1847 1. A plane drawn through a given point is illuminated by two self-luminous spheres; find the position of the plane when the illumination at the given point is a maximum.

2. A small plane touches a self-luminous paraboloid of revolution at its vertex, and is then moved parallel to itself along the axis produced; prove that the illumination of the plane varies inversely as its distance from the focus.

3. Three plane mirrors A, B, C are parallel to the same straight line; find the position of a luminous point, which is so distant that parallax may be neglected, in order that the rays

reflected from A may be parallel to those which are reflected from B and C in succession.

1. * If Q, q be two points in the radius of a spherical re- 1848
fracting surface whose centre is E, such that $EQ : Eq ::$ the sine of incidence : the sine of refraction, determine geometrically the position of the point P so that a ray proceeding from Q and incident upon the surface at P may after refraction proceed from q.

2. * If a ray of light, after being reflected any number of times in one plane, at any number of plane surfaces, return on its former course, prove that the same will be true of any ray parallel to the former which is reflected at the same surfaces in the same order, provided the number of reflections be even.

3. Compare the brightness of the Earth as seen from Venus with the brightness of Venus as seen from the Earth, supposing the sizes and reflecting powers of the two bodies equal.

4. Find the geometrical focus of a pencil of rays refracted through a hollow glass sphere whose external and internal radii are r, r' respectively.

5. Light proceeding from a given point P suffers any number of reflections and refractions ; if consecutive rays of a given colour come out parallel, in the direction determined by the angular co-ordinates, θ, ϕ, shew that $\frac{d\theta}{d\mu} \cdot \frac{d\phi}{d\mu}$ may be obtained by differentiating as if the differently coloured rays which severally come out parallel to their consecutives started from P in the same direction.

Application. In the case of the rainbow of the p^{th} order, given

$$D = p\pi + 2\phi - 2(2p+1)\phi', \sin\phi = \mu \sin\phi',$$

find the order of the colours.

6. Every diameter (d) of the extreme boundary of a spherical reflector subtends a right angle at C the centre of the sphere : supposing parallel rays (inclined at an $\angle$ a to the axis of the reflector) to be incident upon every point of the extreme

boundary, shew that the section of the reflected pencil made by a plane passing through C and perpendicular to the axis will be an hyperbola whose axes are d and $d \cot a$.

1849 1. * If the angle of a hollow cone polished internally be any submultiple of 180°, a cylindrical pencil of rays incident parallel to the axis, will after a certain number of reflexions be a cylindrical pencil parallel to the axis, and of the same diameter as the incident pencil.

2. A ray of light is incident upon one of two reflectors, inclined to each other at an angle $\frac{\pi}{n}$, in a direction parallel to a line which is at right angles to their intersection and bisects the angle between them; supposing the intensity of a ray reflected at an angle ϕ to be to that of the incident ray as $e \cos \phi$ to 1, shew that the intensity of the ray after it has suffered n reflexions will be to that of the incident ray as e^n to 2^{n-1}.

3. A transparent medium is bounded by two parallel planes; the refractive index is constant throughout any plane parallel to the bounding planes, but varies continuously in the direction of the normal to those planes; shew how to find the path of a ray of light through such a medium, and prove that in passing through a section for which the refractive index is a maximum or a minimum, the path will in general have a point of contrary flexure.

PLANE ASTRONOMY.

1838 1. DESCRIBE the sector, and find the largest angles the secant and tangent of which can be found by the sector as usually constructed. Describe the graduation of the lines marked "lines," "inclination of meridians," "latitudes."

2. Shew that if a plane reflector be placed before the object-glass of the telescope of an astronomical circle and nearly perpendicular to the axis of the telescope, an image of the wires of the telescope will be formed in the plane of the wires. Shew

how this property may be employed to determine the nadir point of the circle.

3. If δ, δ', Δ be respectively the greatest, least, and mean apparent diameters of the Moon, and Δ' that at her mean distance; shew that

$$\delta\delta' = \Delta\Delta'.$$

4. If P be the pole, Z the zenith, S the place of a star on the middle wire of a transit instrument, h the hour-angle of the star when on the middle wire, L the east end of a diameter of the sphere parallel to the axis of the transit, then, when the errors of adjustment of the transit are small,

$$\sin ZP \sin SP \,.\, h = \sin SP\,(90° - LZ) + \sin SZ\,(PL - 90°) + \sin PZ\,(SL - 90°).$$

Shew how LZ, PL, SL may be found; and how the error of collimation of a transit instrument may be determined without inversion, by means of a second transit and one meridian mark.

5. The three successive epochs at which a planet's latitude vanishes, and the planet's geocentric longitude at the first and last time its latitude vanished, are observed. Find the true anomaly of the planet at the time of the first observation and the eccentricity of its orbit, the eccentricity being supposed very small.

6. Having given the Greenwich time of the beginning of a given lunar eclipse, and its duration; compare the area on the Earth's surface, including all places which see the whole eclipse with the surface of the Earth considered as a sphere.

1. Assuming the Sun's motion in longitude to be uniform, 1839
find at what season of the year the difference between the apparent lengths of the afternoon and forenoon in latitude 45° will be the greatest possible.

2. If $2h$ be the interval between the Sun's zenith distances being α and β, when his declination is δ at a place whose latitude is l, shew that the values of $2 \cos l \cos \delta \sin h$, $2 \cos l \cos \delta \cos h$,

and $2\sin\frac{a-\beta}{2}\sin\frac{a+\beta}{2}\operatorname{cosec} h$, (taking the upper, middle, and lowest signs, respectively) are equal to

$$\sqrt{\{\cos(l-\delta)-\cos a\}\ \{\cos(l\overset{+}{_+}\delta)\overset{+}{_+}\cos\beta\}}$$
$$\overset{-}{+}\sqrt{\{\cos(l+\delta)+\cos a\}\ \{\cos(l\overset{-}{_-}\delta)\overset{-}{_-}\cos\beta\}},$$

and that the time of rising through a *given* altitude $a-\beta$ will be a minimum, (whether the extreme altitudes a and β, or δ be supposed known) when

$$\tan\frac{a}{2}\tan\frac{\beta}{2}=\tan\frac{l-\delta}{2}\cot\frac{l+\delta}{2};$$

and in the latter case a maximum also when $\delta=\pm\omega$.

If $l<72-\omega$, the duration of twilight will be *greatest* at the solstices; explain why this result cannot be obtained by making $d\delta h=0$.

3. A planet is observed to be stationary when the Earth is 90° distant from the perihelion; having given the eccentricity of the Earth's orbit, the longitude of the perihelion, and the periodic times of the Earth and planet; find the inclination and node of the planet's orbit supposed circular.

4. If (i) be the greatest inclination of a planet's orbit to the ecliptic and (ω) the obliquity, and we assume the planet and its ascending node to be simultaneously in those positions in which their respective declinations are the greatest possible; the planet's latitude (λ) and longitude (l) will be found from the equations,

$$\tan l=-\frac{2\tan\omega}{\sin 2i},\ \text{and}\ \tan\lambda=-\cos l.\tan i.$$

5. Supposing (a) and (b), to be approximate values of two elements whose corrected values are $(a+x)$ and $(b+y)$; V_1, $V_2\ldots V_n$ to be quantities depending by a known law upon these elements and found from (n) observations; U_1, $U_2\ldots U_n$ to be the values which we obtain for V_1, $V_2\ldots V_n$ when we calculate those quantities by means of the approximate elements (a) and

(b): shew how such values of (x) and (y) may be found that, when the quantities V_1, $V_2 \ldots V_n$ are calculated from the corrected elements ($a + x$) and ($b + y$) (neglecting the squares and all higher powers of x and y), the sum of the squares of the differences between these corrected values and the observed values respectively may be the least possible; and determine that least sum.

1. Three stars whose differences of Right Ascension and 1840
distances are known, have the same azimuth at each of two observations taken after a given interval; determine the latitude.

2. Demonstrate formulæ for calculating the time occupied by the disc of the Moon, in any position in the heavens, whilst crossing, first the vertical, and secondly the horizontal, wire of a telescope directed towards it.

3. In a general declining dial the style is bent in its own plane through a given angle α; determine when the angular motion of the Sun's shadow on the dial-plate is a maximum.

1. Prove that the apparent places of the centre of the 1841
moon on the solar disc as seen from different points on the earth's surface, considered spherical, at the same moment of the middle of a given solar eclipse all lie in a straight line which passes through the centre of the solar disc.

2. A semi-elliptical arch stands with its plane vertical and perpendicular to the meridian, and its axis in the plane of the meridian; find the equation to the shadow of the arch on the horizontal plane at six o'clock, in terms of the declination of the sun and the latitude of the place.

Two known stars are seen at a given place A on the same 1842
vertical, when at another place B they are rising together. Find the latitude and longitude of B.

1. If A be the south azimuth of a given vertical plane, c 1843
the colatitude, the north polar distance of a star, h its hour-angle when it crosses the given vertical plane before noon, 2τ

the time (expressed in degress) between its forenoon and afternoon transit across that plane: shew that

$$\tan(\tau - h)\cos c = \cot A, \quad \cos\tau \tan\Delta = \tan c \cos(\tau - h).$$

2. Hence shew, that the latitude of a place, the position of the meridian, and the polar distance of any star may be found, by means of a clock, and two transit instruments placed in vertical planes at right angles to each other. Must we correct for refraction in this method?

How would you place the transits in vertical planes at right angles to each other?

3. The Sun's light is reflected by a mirror to a given point Q. If PN the North Polar Distance of the normal to the mirror be produced to R so that $PR = 2PN$, R traces out a circle round Q.

4. The time of transit of the Moon's diameter across any meridian circle of an equatoreal is nearly unaffected by parallax.

1844 1. The apparent path of a planet on the celestial sphere lies near a certain fixed star; in what manner would you determine, by observation, the time of transit of the planet, across an imaginary disc, of small given angular diameter, described on the celestial sphere with the star as centre?

Shew that the planet's parallax may be found, by observing an imaginary transit of this kind, at two different places properly chosen; supposing, for simplicity, that the longitude of the star differs from that of the Sun by 180°.

2. The altitude and azimuth of a star are observed at given short intervals by means of an altitude and azimuth instrument: shew how a table of refractions might be formed in this manner without knowing the declination of the star or the latitude of the place.

1845 1. The circumpolar portion of the heavens is delineated, according to the principle of Mercator's chart, on the sector of a circle of which the arc represents a given parallel of declination, and the centre represents the pole. Find the radius of the arc which represents any other parallel.

2. The cross wires of an equatoreal are out of adjustment; shew how to find the distance of the intersection of the cross wires from the true centre of the field by observation of a known star.

3. On a given day the curves traced out by the extremities of the shadows of objects of the same vertical height, in whatever latitude they may be, have the same curvature at noon.

1. If λ be the latitude of a place at which, a month before 1846
the autumnal equinox, the day is as long as the longest day at a place in latitude λ', shew that $\tan\lambda = \tan\lambda' \sqrt{1 + 3\sec^2\omega}$ approximately, ω being the obliquity of the ecliptic.

2. In the Italian reckoning of time, half an hour after sunset is designated 24 o'clock: find the hour of noon, at a place in latitude 45°, on a day in August when the tangent of the Sun's declination = sin 15°.

1. Supposing the luminiferous ether to gravitate to the 1847
Sun, and to be incompressible and at rest, and supposing the velocity of light to vary by a small quantity proportional to the pressure of the ether, shew that the observed times of the eclipses of Jupiter's satellites will be affected by an inequality expressed by $c \log \tan \frac{1}{2}(E + S) \cot \frac{1}{4} E$, where c is a constant, and E, S, are two angles of the triangle ESJ.

2. Having given the times by a sidereal clock at which three known stars are observed to have the same zenith distance, the absolute zenith distance being unknown, find the latitude of the place of observation and the error of the clock.

1. If a rectangular court be enclosed within a wall of given 1848
height, and one of its sides be inclined at an angle of 30° to the meridian, determine the breadths of the shadows of the walls on a given day at noon, and the portions of the court and walls which will be enveloped in the shadow, the latitude being 52° 30′ N., and the sun's declination on the given day 7° 30′ N.

2. If there had been no stars how might the absolute periodic times of the Earth and planets have been determined, even if the equator had coincided with the ecliptic.

3. A star map is laid down on the gnomonic projection, the plane of projection being parallel to the equator; give a graphical solution of the problem, to determine the time at a known place by observing when two stars laid down in the map are in the same vertical plane.

4. Shew that at the equinoxes, the extremity of the shadow of the style of a vertical south dial will trace upon the dial plate a horizontal straight line at a distance a cosec l from the upper extremity of the style, a being the length of the style and l the latitude of the place.

1849 1. There are two walls of equal known height at right angles to each other, and running in known directions; shew how to find the Sun's altitude and azimuth by observing the breadth of the shadows of the two walls at any given time. And prove that the sum of the squares of the breadths of the shadows will be the same whatever be the direction of the walls.

2. If the same two stars rise together at two places, the places will have the same latitude. And if they rise together at one place and set together at the other, the places will have equal latitudes, but one North and the other South.

NEWTON'S LUNAR THEORY, &c.

1848 Two bodies P, P' describe round a central body S, circular orbits lying in one plane, the orbit of P being within that of P'; prove that the disturbing force of P' on P, when wholly central, and addititious, will be equal to the disturbing force when P, P' are on opposite sides of S provided SP' be a mean proportional between SP and $SP + SP'$.

1849 If in addition to the force of the Sun on a planet, there be a small force tending towards the Sun and varying inversely as the m^{th} power of the distance of the planet from the Sun, prove that the perihelion of the orbit will have a progressive or regressive motion according as m is greater or less than 2.

Can you explain this result by reasoning similar to that used in "Airy's Gravitation?"

LUNAR THEORY.

SUPPOSE that the Moon moves in the plane of the ecliptic, 1834
that the Sun's apparent orbit is a circle, and that

$$w = a\{1 + m^2 \cos 2(\theta - \theta')\}$$
$$\theta = pt + \tfrac{1}{8} m^2 \sin 2(\theta - \theta'):$$

then shew that the Moon, in addition to the Earth's attraction, is acted on by a force which always passes through the Sun, neglecting terms multiplied by powers of m above the square.

LAPLACE'S COEFFICIENTS AND FIGURE OF THE EARTH.

A HOMOGENEOUS fluid *ellipsoid* of equilibrium revolves uni- 1839
formly round its least axis c: if m be the ratio of the centrifugal force at the extremity of the axis a to the polar gravity g, $a - c = ae$, $b - c = be'$, g' the attraction of the inscribed sphere upon a point at its surface, then

$$3g' = g\left(1 + 2m\frac{c}{a} + \frac{c^2}{a^2} + \frac{c^2}{b^2}\right);$$

find also an equation connecting e and e' as far as terms of the second order.

When the ellipsoid is heterogeneous and the ellipticities of the principal sections small, if G be the gravity at a point whose latitude is λ, in a meridian whose inclination to the plane passing through a and c is l, shew that

$$\frac{G}{g} = 1 - \left\{\left(\frac{5m}{2} - e\right)\cos^2 l + \left(\frac{5m}{2} - e'\right)\sin^2 l\right\}\cos^2\lambda \text{ nearly.}$$

A nearly spherical mass, whose surface is fluid, revolves 1840
permanently about one of the principal axes passing through its centre of gravity; r is the distance of any point at the surface of the fluid from the centre of gravity; θ the angle r makes

with the axis of rotation, ω the angle which the projection of r, upon the plane through the other principal axes, makes with one of them. Prove that if r be expanded in a series of Laplace's coefficients, in the form

$$a\,(1 + a\,Y^{(0)} + a\,Y^{(1)} + a\,Y^{(2)} + a\,Y^{(3)} + \ldots\ldots),$$

then $Y^{(1)} = 0$, and $Y^{(2)} = k\,(\cos^2\theta - \frac{1}{3}) + k'\,\sin^2\theta\cos 2\omega$, where k, k' are constants.

1843 If $\dfrac{1}{\sqrt{(1 - 2\mu h + h^2)}} = P_0 + P_1 h + P_2 h^2 + \ldots P_n h^n + \ldots$, shew that

$$\frac{d}{d\mu}\left\{(1 - \mu^2)\,\frac{dP_n}{d\mu}\right\} + n\,(n+1)\,P_n = 0, \quad \frac{d^n(P_n)}{d\mu^n} = 1\,.\,3\,.\,5\ldots(2n-1),$$

and if $P_n = C\,(\mu^n + A_1\mu^{n-1} + A_2\mu^{n-2}\ldots)$ determine C, A_1, A_2,...

PRECESSION AND NUTATION.

1843 1. If the Earth had no motion of rotation about its axis, but yet preserved the form of an oblate spheroid, the action of the Sun would cause a uniform progression of the mean equinox, and the true equinox would make small oscillations about the mean, whose period would be three months nearly.

2. The Moon's libration in longitude (u) is given by the equation

$$\frac{d^2u}{dt^2} + 3m^2\frac{B - A}{C}\,u = 3m^2\frac{B - A}{C}\,\Sigma H \sin(ht + h').$$

Integrate this equation, and examine the effect of the several terms. Shew that observations of (u) might afford a means of determining $\dfrac{B - A}{C}$.

ATTRACTIONS.

1. A particle is attached by means of a string to the center of a hemispherical shell of attractive matter; determine the tension of the string, supposing its length less than the radius of the shell, the force of attraction to vary as $\dfrac{1}{d^2}$.

2. Two equal straight rods whose particles attract each other with a force $\propto \frac{1}{d^2}$ are parallel to each other and perpendicular to the lines joining their ends, and prevented from coming together by strings attached to their middle points: determine the tension of the strings when the rods are at a given distance from each other.

If F, G, H be the attractions, in the directions of the axes, 1839
of an ellipsoid upon an *external* point (f, g, h); V the volume of a concentric ellipsoid passing through that point and having its principal sections confocal with those of the first ellipsoid, then $V\left(\frac{F}{f} + \frac{G}{g} + \frac{H}{h}\right)$ is constant for all external points; prove this, and determine its value.

If θ be the angle between any two lines in space which 1840
attract each other with forces varying as $\frac{1}{D^2}$, the whole attraction in the direction of the shortest line between them, is $= \frac{2\pi}{\sin\theta}$; the mutual attraction of two units of length, collected in centres and separated by the unit of distance, being considered equal to unity.

A slender ring DEF is attached to another ring ABC by 1842
means of a string AD whose length = the radius of ABC; supposing DEF to fall entirely within ABC, determine the tension of the string, when DEF is at rest, force of attraction of ABC $\propto g \frac{1}{(\text{dist.})^3}$.

The attraction of an ellipsoidal shell on an external particle 1843
is in the direction of the axis of the enveloping cone, whose vertex is the given particle.

A very small bar of matter is moveable about one extremity 1845
which is fixed half-way between two centres of force attracting inversely as the square of the distance; if l be the length of the bar, and $2a$ the distance between the centres of force, prove

that there will be two positions of equilibrium for the bar, or four, according as the ratio of the absolute intensity of the more powerful force to that of the less powerful, is or is not greater than $\frac{a+2l}{a-2l}$: and distinguish between the stable and unstable positions.

1848 1. A sphere is composed of an immense number of small free particles, equally distributed, which gravitate to each other without interfering; supposing the particles to have no initial velocity, prove that the mean density about a given particle will vary inversely as the cube of its distance from the centre.

2. Prove geometrically, or otherwise, that if g be the attraction which a particle m exerts on a point in a closed surface S, θ the angle between the direction g and the normal, $d\omega$ an element of S,

$$\iint g \cos\theta \, d\omega = 4\pi m, \text{ or } = 0,$$

according as m is within or without S, the attraction of m at the distance r being $\frac{m}{r^2}$.

Extend this result to the case of a finite mass cut by S, and thence prove, by taking for S an elementary parallelopiped, that if V be the potential of any mass for an internal particle.

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = -4\pi\rho.$$

3. Supoosing a mass of homogeneous fluid which attracts every particle of matter with a force varying as $\frac{1}{(\text{dist.})^2}$ to be enclosed within a thin spheroidal shell, find the path described by a heavy body let fall from any point of the surface of the fluid, the resistance of the fluid varying as the velocity. Prove also that the body will reach the axis and equator of the spheroid after the same intervals respectively from whatever point of the surface it begins to fall.

1849 The attraction of a uniform filament of matter, in the form of a plane curve, upon a particle, is replaced by that of a

circular filament having the particle for its centre; find the law of density of the circular filament in order that this may be done.

PHYSICAL OPTICS.

1. A BEAM of homogeneous light is transmitted through 1838
a slice of a uniaxal crystal bounded by planes parallel to each other and to the optic axis, in a plane through the optic axis perpendicular to the surfaces of the slice. Find the difference of retardations of the ordinary and extraordinary emergent rays.

2. A telescope the object-glass of which is covered by a screen containing two equal circular holes, is directed to a point of homogeneous light at an indefinitely great distance; having given the intensity of the light in any part of the field of view when one hole is closed, find the intensity when both holes are open, and shew that the field of view will be crossed by black bands perpendicular to the line joining the centers of the holes, and that the distance between any two consecutive lines will be inversely as the distance between the centers of the holes.

3. Three prisms are cut out of a biaxal crystal with their edges respectively parallel to the three axes of elasticity: having given the angles of each prism and the minimum deviations of a ray of homogeneous light refracted through it, polarized in a plane perpendicular to the edge of the prism, find the angles which the optic axes make with the axes of elasticity, and if a slice of the crystal be bounded by planes perpendicular to that axis of elasticity which bisects the acute angle between the optic axes, find the angle which the optic axes seen through it, appear to make with each other.

On the emission theory of light the amount of aberration is 1840
independent of the density of the medium through which the light comes to the eye, but not so on the undulatory theory.

1. If $\alpha\beta\gamma$ be the displacements, parallel to the axes of xyz, 1843
of any particle of the luminiferous ether from its equilibrium position (xyz), and if

$$\alpha = a\cos\phi + (qc - sb)\sin\phi, \quad \beta = b\cos\phi + (sa - pc)\sin\phi,$$
$$\gamma = c\cos\phi + (pb - qa)\sin\phi, \quad pa + qb + sc = 0, \quad p^2 + q^2 + s^2 = 1,$$
$$\text{and } \phi = k(vt - px - qy - sz),$$

$abcpqsk$ and v being constants: shew what each of these constants represents, and prove that the light is circularly polarized.

2. A ray of light polarized in the plane of incidence enters a crystal cut perpendicularly to its axis, and is separated into two elliptically polarized rays of opposite but equal ellipticities, find the intensity after reflection from an analyzing plate.

1845 Account for halos and mock Suns. Find the greatest altitude of the Sun at which the latter can be seen.

1846 1. Show that according to the common hypotheses made in the corpuscular theory of light the refractive index of a given substance, for light coming from a given star, ought to be increased by the motion of the Earth by the quantity $\frac{a}{\mu}(\mu^2 - 1)\cos\lambda\sin(l - s)$ nearly, where a is the coefficient of aberration, λ the latitude of the star, s its longitude, and l the longitude of the Earth.

2. A biaxal crystal presents naturally two plane faces perpendicular to two of the axes of elasticity; the crystal being immersed in fluid, a pencil of homogeneous light is transmitted through it across those faces, in a plane perpendicular to them both, and the minimum deviation (D) of the extraordinary emergent pencil is observed. Show that the observation furnishes towards the determination of the principal indices of refraction the equation

$$(\mu_1^2 - \mu^2)(\mu_2^2 - \mu^2) = \mu^4 \sin^2 D,$$

where $\mu_1\left(=\frac{v}{a}\right)$, $\mu_2\left(=\frac{v}{b}\right)$ are two of the principal indices required, and μ is the refractive index of the fluid.

1847 1. A ray of light passes from air into a transparent prism of either single or double refraction, and emerges after any number of internal reflections; apply the principle of quickest propagation to prove that the emergent ray or rays are inclined to the edges of the prism at the same angle as the incident ray.

2. The refractive index (μ) of a transparent medium varies continuously from point to point; prove by means of the principle of quickest propagation, or otherwise, that the differential equations to a ray of light within the medium are

$$\frac{ds}{dx}\left(\frac{d\mu}{dx} - \mu\frac{d^2x}{ds^2}\right) = \frac{ds}{dy}\left(\frac{d\mu}{dy} - \mu\frac{d^2y}{ds^2}\right) = \frac{ds}{dz}\left(\frac{d\mu}{dz} - \mu\frac{d^2z}{ds^2}\right),$$

where the differential co-efficients of μ are partial.

A spherical wave of light is incident directly on a lens; find 1848
approximately the retardations of the several portions of the wave, and prove in this way the common equation $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$.

SOUND, VIBRATIONS.

1. The vibrations of an elastic medium being defined by 1845
the equations

$$\frac{a}{a} = \frac{\beta}{b} = \frac{\gamma}{c} = F(vt - lx - my - nz),$$

where $x + a, y + \beta, z + \gamma$ are the co-ordinates at the time t of a molecule whose co-ordinates in the state of rest are x, y, z, and a, b, c are certain constants; the vibrations will be transversal to the direction of propagation if

$$\frac{da}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 0.$$

2. Two pipes, closed at one end and open at the other, one of which is slightly longer than the other, but which are in all other respects exactly similar, are placed side by side, and made to sound their fundamental notes; prove that the resultant sound will vary from silence to twice the intensity of either of the pipes sounded singly, and the interval between two successive silences will be $\frac{4}{a}\frac{ll'}{l' - l}$, where a is the velocity of sound, and ll' the lengths of the pipes. It may be assumed that the type of aerial vibrations is $c \sin\frac{2\pi}{\lambda}(at - x)$.

1. Determine the scale of harmonics which can be pro- 1846
duced by a stretched string.

2. $x'Ax$ is a stretched string, of which the portions Ax, Ax' are of different thicknesses; prove that any small transversal vibrations travelling from x to A will, on arriving at A, be partly reflected and partly transmitted to Ax', and that the displacements due to the incident, reflected and transmitted vibrations are to each other as $1 + \mu : 1 - \mu : 2$, where μ is the ratio of the velocity of propagation in Ax to that in Ax'.

1849 Supposing the effect of friction in the case of aerial vibrations in a tube of uniform bore to be the production of a retarding force on each particle equal to $f \times$ velocity, prove that the equation of motion will be satisfied by taking

$$ce^{-\frac{ft}{2}} \sin \frac{2\pi}{\lambda} \left\{ a\left(1 - \frac{f^2\lambda^2}{16a^2}\right)^{\frac{1}{2}} t - x \right\}$$

as the type of the vibrations.

ELECTRICITY AND MAGNETISM.

1838 A SOLID is generated by the revolution of the curve $r = a(1 + \epsilon \cos^2\theta)$ about the axis from which (θ) is measured; determine the law of distribution of free electricity on its surface, and compare the accumulation at points corresponding to $\theta = 0$ and $\theta = \frac{\pi}{2}$, supposing (ϵ) very small.

1839 When electricity is latent in a slender straight rod, and the *type* is taken for the form of the distribution, there are an indefinite number of points very near each other at which the electricity is neutral; it is arranged in successive cumuli of alternately positive and negative electricities; and the accumulation at points equidistant from the extremities are equal. Compare also the accumulations at the middle and extremities of the rod.

1840 If a very *short* fixed rectilinear magnet, one end of which has just as much repulsive as the other has attractive power, act upon a *small* needle suspended freely by one extremity, and $(90 - \lambda)$ be the angle which the magnet makes with the line drawn from its centre to the fixed extremity of the needle:

shew that the intensity of the force exerted to turn the needle will be proportional to $\sqrt{1 + 3\sin^2\lambda}$, and the dip ($\delta$) be found from the equation $\tan\delta = 2\tan\lambda$.

Shew also that if the Earth's magnetism arose from the action of *any number* of small magnets concentric with it and with each other, it would have only one magnetic axis, lines of equal dip and of equal intensity would coincide and both be parallels of magnetic longitude, and the greatest intensity would be twice the least.

Prove that, if at each point of space a force act which is any 1841
function of its distance from a given point A, and θ be the angle at which the tangent to a point P of an arbitrary curve, connecting any two points P_1 P_2 in space, is inclined to the direction of the force f at P, then $\int f\cos\theta\, ds$, from P_1 to P_2, depends only on the distances AP_1, AP_2.

Assuming that the law of force in the action of the particles of the magnetic fluid is that of the inverse square of the distance, shew that the existence of two points on the earth's surface where the total horizontal magnetic force vanishes, and the total intensity is a maximum for both or a minimum for both, necessitates the existence of a third point where the total horizontal force vanishes.

The resultant of the attraction of a very small magnet on a 1844
particle of free magnetism passes through a given point. The position of the magnet being given, determine the curve on which the particle must lie.

1. Two weights are successively fixed on the southern arm 1845
of a dipping needle; having given the moments of the weights and the inclination of the needle to the horizon in the positions of equilibrium, find the true dip.

2. If the horizontal magnetic force of the Earth at any point on its surface be resolved into two, X in the direction of the geographical meridian, and Y in that perpendicular to it, and X be given generally as a function of the latitude and longitude, then Y may be fully determined; but if Y be given, the same proposition cannot be asserted concerning X.

3. Find the position of a small rectilinear magnet, the centre of which is fixed, when the action upon a distant particle of free magnetism is in a given direction.

4. Given the inclination of a small horizontal bar magnet to the magnetic meridian, find the position in which the centre of a small horizontal needle must be placed, in order that the action of the magnet may not cause the needle to deviate from the magnetic meridian, supposing the two magnets to be in the same plane.

HEAT.

1844 1. It has been proposed to represent the mean diurnal changes of temperature by such a formula as

$$t = A_0 + A_1 \cos\phi + \ldots A_p \cos p\phi + B_1 \sin\phi + \ldots B_p \sin p\phi,$$

where ϕ is the hour-angle. Shew that if the mean of the temperatures observed through twenty-four hours at m equidistant intervals of time, is not the same at whatever time the series of observations commences, it is necessary to suppose p to be greater than $m - 1$.

2. The Sun's rays are allowed to pass through a plate of glass, and to fall on the blackened bulb of a thermometer; shew how, by varying the circumstances of the experiment, the effect due to radiation from the surface of the glass may be eliminated. Why is the Sun's radiated heat felt more under glass than in still air?

PART II.

EUCLID.

1. If one side of a triangle be produced, the exterior angle 1845
is greater than either of the interior opposite angles.

2. In equal circles equal circumferences are subtended by equal straight lines.

3. Define compound ratio.

Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.

4. Draw a straight line perpendicular to a plane from a given point above it.

1. Define a parallelogram, and shew that the opposite sides 1846
and angles of parallelograms are equal.

2. Prove that the opposite angles of any quadrilateral figure which can be inscribed in a circle are together equal to two right angles. State whether the converse is true.

3. Equal triangles, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional.

4. If two planes which cut one another be each of them perpendicular to a third plane, their common section shall be perpendicular to that third plane.

1. The opposite sides of a parallelogram are equal to one 1847
another, as are also the opposite angles, and the parallelogram itself is bisected by its diagonal.

2. The angles in the same segment of a circle are equal to one another.

3. Similar triangles are to each other in the duplicate ratio of their homologous sides.

4. Draw a straight line perpendicular to a plane from a 2 given point above it.

1848 1. * If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides, equal to them, of the other.

* How does it appear that the two triangles are equiangular and equal to each other?

2. * The opposite sides as well as the opposite angles of a 3 parallelogram are equal to one another, and the diameter bisects it.

3. * Equal straight lines in a circle are equally distant from 1 the centre; and conversely, those which are equally distant from the centre are equal to one another.

4. * The angle at the centre of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference.

5. * Describe an isosceles triangle having each angle at the base double of the third angle.

6. * If the first of six magnitudes have to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth to the sixth, the first shall have to the second a greater ratio than the fifth has to the sixth.

7. * Equal triangles which have one angle of the one equal 2 to one angle of the other have the sides about the equal angles reciprocally proportional; and conversely.

8. * If two planes which cut each other be each of them per-

pendicular to a third plane, their common section shall be perpendicular to the same plane.

9. If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

1. *Describe an equilateral triangle upon a given finite straight line. 1849

2. *If a side of any triangle be produced, the exterior angle is equal to the sum of the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

3. *Divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.

4. *If a straight line touch a circle, the straight line drawn from the centre to the point of contact shall be perpendicular to the line touching the circle.

5. *Inscribe a circle in a given triangle.

6. *If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

*What restriction is here implied as to the species of the magnitudes?

7. *The sides about the equal angles of equiangular triangles are proportionals, and those sides which are opposite to the equal angles are homologous.

8. Define compound ratio; and prove that equiangular parallelograms have to each other the ratio which is compounded of the ratios of their sides.

*Of what use is this proposition in the application of Algebra to Geometry?

9. *Draw a straight line perpendicular to a plane from a given point without it.

ALGEBRA.

1845 1. STATE and explain the rule for the extraction of the square root of an algebraical expression.

2. When is one quantity said to vary directly as another? If x varies directly as y when z is constant, and inversely as z when y is constant, then if y and z both vary x will vary as $\frac{y}{z}$.

3. Assuming the expansion of $(1+x)^n$, n being a positive integer, determine the relation between any two consecutive terms of $(1+x)^{-n}$.

If $\left(\frac{a}{a-x}\right)^n$ be expanded in a series of ascending powers of x, at what term will the series begin to converge, x being less than a?

4. Shew how to find the number of positive integral solutions of $ax + by = c$, where a, b and c are integers.

5. Investigate a rule for transforming a number from one scale of notation to another.

6. Prove that a recurring series may in general be resolved into a number of geometrical series. What exception is there to this proposition?

7. If p be a prime number and a be not divisible by p, then a^{p-1} is of the form $mp + 1$; and the index of the lowest power of a which has this form is either $p - 1$ or a submultiple of it.

1846 1. Shew how to transform a number from one scale of notation to another.

2. If a and b be two integral numbers prime to one another, and the product $a \times c$ be divisible by b, shew that c must be divisible by b.

Find the form of the denominator of a vulgar fraction in its lowest terms, when it is reducible to a terminating decimal. Is $\frac{51}{120}$ so reducible?

3. Prove the formula for the number of variations of n things taken r together.

4. Find the greatest term in the expansion of $(1 \pm x)^n$ without regard to sign, n being positive, and x a proper fraction. Will the same investigation apply when n is negative?

5. Find the sum of an arithmetical series of n terms.

6. Find the amount of a sum of money in any number of years at Compound Interest.

7. Define the characteristic of a logarithm, and give those of $\log_{10}1230$ and $\log_{10}.0123$. Assuming the series for the expansion of $\log(1+x)$ express the logarithm of any number in terms of the logarithms of the two preceding numbers, and of a rapidly converging series.

8. Shew that the series $u_0 + u_1 + \ldots + u_n + \ldots$ will be convergent if the greatest of the limits of $(U_n)^{\frac{1}{n}}$, when n is indefinitely increased, be less than unity, and divergent if it be greater, U_n being the numerical value of u_n.

Prove also that if, as n is indefinitely increased, $\frac{U_{n+1}}{U_n}$ converge to a finite limit, then $(U_n)^{\frac{1}{n}}$ will converge to the same limit.

1. If m be the least common multiple of a and b, the least 1847
common multiple of m and c is the least common multiple of a, b and c.

2. Find the sum of (n) terms of an Arithmetic series.

3. A railway train after travelling for 1^h meets with an accident, which delays it 1^h, after which it proceeds at $\frac{3}{5}$ths of its former rate, and arrives at the terminus 3^h behind time; had the accident occurred 50 miles further on, the train would have arrived $1_h\ 20^m$ sooner; required the length of the line.

4. Find the number of variations which can be formed out of n things taken all together, when p are of one sort, q of another, &c.

5. There are n of each of the m letters $a, b, c \ldots$; find the

whole number of variations which can be made of them without taking more than n letters together.

6. Find the present value of a given half-yearly payment to continue for any number of years, at a given rate per annum, compound interest.

7. Shew that the number of positive integral solutions of the equation $ax + by = c$ cannot differ by more than unity from the greatest integer contained in $\frac{c}{ab}$.

8. Define a logarithm, and prove that

$$\log xy = \log x + \log y, \log x^n = n \log x, \frac{\log_a x}{\log_b x} = \log_a b = \frac{1}{\log_b a}.$$

9. If $a, b, c \ldots$ are not proportional to $\alpha, \beta, \gamma \ldots$, prove that the ratio of $a + b + c + \ldots$ to $\alpha + \beta + \gamma + \ldots$ lies between the greatest and the least of the ratios $a : \alpha$, $b : \beta$, $c : \gamma$ .....

1847 1. Explain the meaning and use of fractions in a system of arithmetic, and shew that the value of a fraction remains unchanged when its two members are replaced by any equimultiples of their former values.

2. Prove the rules for finding the greatest common measure and least common multiple of two integers.

3. Supposing arithmetical addition and subtraction represented by the signs +, and − respectively, prove the equivalence of the two sets of operations indicated by $(a - b)(c - d)$ and $ac + bd - bc - ad$, a being a number greater than b, c greater than d.

* What is the nature of the generalization with respect to the use of the signs +, − in Symbolical Algebra?

4. * Define what is meant by the n^{th} root of a given numerical magnitude, and explain the principle according to which a root is represented by means of a fractional index.

5. * When is one quantity said to vary as another, directly or inversely? If $A \propto B$ when C is constant, and $A \propto C$ when B is constant, prove that $A \propto BC$ when B and C both vary.

6. * Find the sum to n terms of a geometric series. What is meant by the sum of an infinite series? When can such a series be said to have a sum?

7. * Assuming the Binomial Theorem in the case of positive integral indices, prove it in the case of negative and fractional indices.

8. Investigate an expression for the general term in the expansion of the polynomial $(a_0 + a_1x + a_2x^2 + \ldots + a_mx^m)^n$, n being a positive integer.

9. If a is prime to b and c, it is prime to their product.

Prove also that a^n is prime to b^n, and hence shew that $\sqrt[n]{N}$ is incommensurable, N being a whole number not a perfect n^{th} power.

10. If a quantity P be expressed as a continued fraction, shew that any two consecutive converging fractions $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$ are one greater and the other less than P, and that the difference between $\frac{p_2}{q_2}$ and P is less than $\frac{1}{q_2^2}$.

1. * Assuming generally that $a^m \cdot a^n = a^{m+n}$, determine the meaning which must be assigned to a^m when m is fractional or negative. 1849

2. * Prove the rule for extracting the square root of an integer.

3. * Shew that a ratio of greater inequality is diminished by adding the same quantity to both terms.

4. * Find the number of combinations of n things taken r together; and shew that the number of combinations of n things taken r together is equal to the number taken $n - r$ together.

5. * Prove the rules for the multiplication and division of decimals. Shew that every vulgar fraction must produce either a terminating or a recurring decimal.

6. * If $\frac{a}{b}$ be a fraction, and if a and the successive remainders

be multiplied by a series of quantities q, q', q'', &c., and the successive products be divided by b, giving quotients p, p', p'', &c. shew that

$$\frac{a}{b} = \frac{p}{q} + \frac{p'}{qq'} + \frac{p''}{qq'q''} + \&c.$$

7. Investigate the expansion of a^x in a series proceeding by powers of x and find at what term the series begins to converge.

8. The probability of an event's happening in one trial being given, find the probability of its happening at least t times in n trials.

9. Sum the series $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3$; and prove that when n is indefinitely increased, the fraction $\frac{1^p + 2^p + 3^p + \ldots + n^p}{n^{p+1}}$ approaches to $\frac{1}{p+1}$ as its limit.

THEORY OF EQUATIONS.

1845 1. If a, b, c, &c. are the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots = 0,$$

shew that $a^2 + b^2 + c^2 + \&c. = p_1^2 - 2p_2$.

2. Prove that an algebraical equation, whose roots are not all real, must have an even number of impossible roots.

3. An odd number of real roots of the equation $f'(x) = 0$ lies between each adjacent pair of real roots of $f(x) = 0$.

4. When all the real roots of $f'(x) = 0$ can be obtained, shew how to determine the number and situation of the real roots of $f(x) = 0$.

1846 1. Prove that an odd number of real roots of the equation $f'(x) = 0$ lies between each adjacent pair of real roots of $f(x) = 0$.

If all the real roots of $f'(x) = 0$ are known, shew how to find the number and situation of the real roots of $f(x) = 0$.

2. If an equation has all its roots imaginary, shew how to find when the product of the moduli of the roots is 1.

3. Give Euler's method of solving a biquadratic equation.

1. If $f(x)$ is any rational and integral function of x, prove 1847
that the highest common factor of $f(x)$ and $f'(x)$ consists of the product of the simple factors which occur more than once in $f(x)$, each raised to a power one less than that with which it occurs in $f(x)$.

2. If the coefficients of an equation be rational, surd roots of the form $a + \sqrt{b}$ enter by pairs.

3. Investigate a method of finding the commensurable roots of an equation whose coefficients are rational.

4. Given the roots of a biquadratic equation, find those of the reducing cubic which occurs in its solution by the method (1) of Euler, (2) of Descartes.

1. Every equation has at least one real root, unless it be of 1848
an even degree with its last term positive; and if it have only one change of sign, it has only one positive root.

2. If p be the greatest coefficient of an equation, without regard to sign, and q the last term, shew that all the real roots lie between $p + 1$ and $-(p + 1)$.

3. Prove that a reciprocal equation of $2n + 1$ dimensions may be solved by means of an equation of n dimensions; and a reciprocal equation of $2n$ dimensions may be solved by means of an equation of n or $n - 1$ dimensions, according as the last term is positive or negative. Also obtain the equation upon which the solution is made to depend.

1. If $f(a)$ and $f(b)$ have different signs, an odd number 1849
of roots of the algebraical equation $f(x) = 0$ lies between a and b.

2. Explain Horner's method of approximating to the real roots of algebraical equations.

3. Express the sum of the m^{th} powers of the roots of an equation in terms of the coefficients and the sums of inferior powers.

PLANE TRIGONOMETRY.

1845 1. PROVE that

$$\tan(\theta + \theta') = \frac{\tan\theta + \tan\theta'}{1 - \tan\theta \tan\theta'},$$

2. The positions of three stations A, B and C, have been laid down on a map, and an observer at D (a station in the same horizontal plane as A, B and C) determines the angles ADB and BDC; give a geometrical construction for laying down D on the map.

3. Assuming the formula $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, find an expression for the area of a plane triangle in terms of the sides.

4. Expand $(\cos\theta)^n$ in a series of cosines of multiple arcs, n being a positive integer.

From the result deduce the expansion of $(\sin\theta)^n$.

5. Prove that $(\cos\theta + \sqrt{-1}\sin\theta)^m = \cos m\theta + \sqrt{-1}\sin m\theta$, whether m be integral or fractional, positive or negative.

6. Shew how to calculate the three roots of a cubic equation by means of trigonometrical tables, the roots being all real.

1846 1. Express $\tan A$ in terms of $\tan 2A$, and shew, *a priori*, why two values must be expected.

2. Shew that $\sin(A - B) = \sin A \cos B - \cos A \sin B$, when A is greater than B, and A and B between 0 and 90°; and thence shew that the same equation is true, when both A and B are between 90° and 180°.

3. Explain the ambiguous case in the solution of plane triangles.

4. If two sides a and b of a plane triangle and the included angle C be given, and b be very small compared with a, the angle B may be found from the formula,

$$B'' = \frac{b}{a}\frac{\sin C}{\sin 1''} + \frac{b^2}{a^2}\frac{\sin 2C}{\sin 2''} + \ldots.$$

5. Every quantity of the form $a + \sqrt{-1}\,b$, where a and b are real, may be reduced to the form $\rho \cdot (\cos\theta + \sqrt{-1}\,\sin\theta)$, where ρ is positive.

6. Prove that

$$\sin n\theta = 2^{n-1} \cos\theta \sin\theta \left(\sin^2\frac{\pi}{n} - \sin^2\theta\right)\left(\sin^2\frac{2\pi}{n} - \sin^2\theta\right)\ldots$$

$$\left(\sin^2\frac{n-2}{2}\frac{\pi}{n} - \sin^2\theta\right),$$

where n is even, and thence deduce the theorem

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\ldots$$

1. Express $\sin A$ and $\cos A$ in terms of $\sin 2A$, explaining 1847
the mode of determining the signs of the radicals in any particular case.

2. Express $\sin A \pm \sin B$ and $\cos A \pm \cos B$ in products involving the sines or cosines of $\frac{1}{2}(A + B)$ and $\frac{1}{2}(A - B)$.

3. Prove the formula $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$, and apply it to solve a triangle when two sides and the included angle are given.

Point out an objection to the use of this formula in the case in which a and b are nearly equal, and give another method of solution to which this objection does not apply.

4. Investigate the exponential expressions for $\sin\theta$, $\cos\theta$ and $\tan\theta$. State what is taken for the unit of angular measure in these expressions, and apply them to prove that

$$x\sin\theta + \frac{x^2\sin 2\theta}{1.2} + \frac{x^3\sin 3\theta}{1.2.3} + \ldots = \epsilon^{x\cos\theta}\sin(x\sin\theta).$$

5. The general term in the expansion of $\cos^m\theta \cdot \cos^n\phi$, m and n being positive integers, in terms of the cosines of the multiple angles is

$$\frac{{}^mC_r \cdot {}^nC_s}{2^{m+n}} \cdot \cos\{(m - 2r)\,\theta + (n - 2s)\,\phi\},$$

where r and s may be any integers from 0 to m, and from 0 to n respectively.

1. Trace the change in sign and magnitude of the tangent 1848
and secant of an angle through the first four quadrants.

2. *Prove that $\sin(A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B$.

Give a general enunciation for the construction requisite in order to obtain the above expression for all values of A and B.

3. *Express $\sin 2A$ in terms of $\tan A$.

4. Prove that the signs of the angles of a triangle are proportional to the opposite sides.

*Hence deduce the expression for the cosine of an angle in terms of the sides.

5. *Express the area of a triangle in terms (1) of two sides and the contained angle, (2) of one side and the adjacent angles.

6. *Shew how to determine the height of a mountain by observations at two stations in the same horizontal plane, the distance between the stations being known.

7. *Define a logarithm, and the base of a system of logarithms.

Prove that $\log xy = \log x + \log y$, $\log x^n = n \log x$.

8. When two angles are nearly equal, shew that in general the difference between their logarithmic cotangents is nearly proportional to the difference between the angles; and point out the cases in which this statement will be incorrect.

9. Assuming De Moivre's theorem in the case of positive integral indices, apply it to obtain the expansions of $\sin x$ and $\cos x$ in series ascending according to powers of x, proving that the limit of $\left(\cos \frac{\theta}{n}\right)^n$ for $n = \infty$ is 1.

1849 1. *Define the cosine of an angle; and trace its changes in sign and magnitude as the angle increases from 135° to 405°.

2. *Express $\sin A$ in terms of $\sin 2A$; explain why there are four values, and how the correct one is to be selected.

3. *Two sides and the included angle of a triangle being given, shew how to find the remaining angles.

4. *Find the area of a quadrilateral figure, whose opposite angles are supplementary, in terms of its sides.

5. *Shew how to find the height of an object above a horizontal plane from observations made at two given stations in the plane.

6. Define a logarithm; and find that of 256 to the base $2\sqrt{2}$.

What is the advantage of taking 10 for the base of a system of logarithms?

Calculate a table of proportional parts for every 10″, corresponding to a tabular difference of 4797 for 1′.

7. Express the circular measure of an angle in a series ascending by powers of its tangent.

How may the above series be used for calculating the value of an angle that lies between $\frac{\pi}{4}$ and $\frac{\pi}{2}$, when the tangent of the angle is given?

SPHERICAL TRIGONOMETRY.

1. In spherical triangles the sines of the angles are pro- 1845
portional to the sines of the opposite sides.

2. The area of a spherical triangle varies as the excess of the sum of its angles over two right angles.

3. Given the logarithms of two consecutive whole numbers, $p, p+1$, investigate a series for the calculation of $\log(p+2)$.

If the tabular logarithms of two consecutive numbers of considerable magnitude be given, find the Napierian logarithms of the same numbers.

4. The number of faces of any polyhedron, added to that of its corners, exceeds the number of its edges by two.

5. Prove Legendre's theorem for solving a spherical triangle, the lengths of the sides of which are small compared to the radius of the sphere.

1. A is an angle of a spherical triangle, a' the corresponding 1846
side of the polar triangle; prove that $A + a' = 180^0$.

2. Prove Napier's rules when one of the sides is the mid-

dle part, and shew that in any triangle ABC, $\tan\frac{A}{2} = \frac{\tan\rho}{\sin(s-a)}$, where ρ is the radius of the inscribed circle, and $2s$ is the perimeter.

3. From a given formula in Spherical Trigonometry, shew how to deduce the corresponding formula in Plane Trigonometry. What formulæ in Plane Trigonometry are deducible from Napier's analogies?

4. What difficulty presents itself in the calculation of the spherical triangles used in geodetical operations, and by what different methods is it overcome?

5. If b, c are two sides of a very small spherical triangle and $A_,$ the angle of the chordal triangle corresponding to A, prove the approximate formula

$$A_, = A - \sin^2\tfrac{1}{4}(b+c)\tan\frac{A}{2} + \sin^2\tfrac{1}{4}(b-c)\cot\frac{A}{2}.$$

1847 1. Express the cosine of an angle of a spherical triangle in terms of the sides.

2. Given the base and vertical angle of an isosceles spherical triangle, find the equal sides.

3. In geodetic measurements what are the angles which are respectively observed with a repeating circle and with a theodolite? Investigate an approximate formula for obtaining the latter angles from the former. Point out the difficulty of solving the triangles as spherical triangles, and a method by which such a solution may be effected.

1848 1. Having given two sides a, b of a spherical triangle, and the angle A opposite to one of them, obtain an equation for determining the angle C; and adapt the formula to logarithmic computation.

2. In a spherical triangle, prove that

$$\sin\frac{a-b}{2} = \frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}}\sin\frac{c}{2};$$

$$\cos\frac{a-b}{2} = \frac{\sin\frac{A+B}{2}}{\cos\frac{C}{2}}\cdot\cos\frac{c}{2}.$$

1. Prove that in any spherical triangle 1849

$$\cot a \sin c = \cot A \sin B + \cos c \cos B.$$

2. Investigate Napier's rules for solving a right-angled spherical triangle, when one side is taken for the middle part.

3. About the angular points of a regular polyhedron of given species as centres, spheres are described each with radius equal to half the length of an edge; find the radius of the sphere which circumscribes them all.

CONIC SECTIONS.

1. DEFINE a parabola, and its tangent. 1848

* Assuming the tangent at any point P of a parabola to make equal angles with the focal distance SP and the diameter at that point, prove that SY, the perpendicular upon it from the focus, meets it in the tangent at the vertex.

2. Prove that in a parabola $QV^2 = 4SP \cdot PV$.

* Define the parameter at any point of a parabola, and prove that it is proportional to the focal distance of the point.

3. * Define an ellipse. If one of the focal distances SP of a point P be produced to L, a straight line PT which bisects the exterior angle HPL is the tangent to the curve at P.

4. * State when diameters of an ellipse or hyperbola are conjugate. Prove that all parallelograms whose sides touch an ellipse at the ends of conjugate diameters are equal.

5. * In an ellipse the sum of the squares of any two conjugate diameters is invariable.

6. * Define the asymptotes of an hyperbola. If any straight line Qq perpendicalar to either axis of an hyperbola meet the asymptotes in Q, q, and the curve in P the rectangle $QP \times Pq$ is invariable.

7. * If a circular right cone be cut by a plane which meets both of its slant sides, the section is an ellipse.

8. * The chord of curvature of an ellipse or hyperbola through its centre is equal to $\frac{2\,CD^2}{CP}$.

1849 1. * In the parabola, the rectangle under the latus rectum and an abscissa of the axis is equal to the square of the semi-ordinate.

2. * The normal at any point of an ellipse bisects the angle between the focal distances.

* Can you deduce the proof of this proposition from mechanical considerations?

3. The perpendiculars from the foci on the tangent to an ellipse intersect the tangent in the circumference of a circle having the axis major as diameter.

* Deduce from this an analogous proposition for the parabola.

4. * In the ellipse, if the conjugate diameter meet either focal distance in E, PE will be equal to AC.

5. * Define the circle of curvature; and prove that in the ellipse the diameter, the conjugate diameter, and the chord of curvature passing through the centre, are in continued proportion.

6. If a tangent be drawn to a hyperbola, and be terminated by the asymptotes, it will be bisected in the point of contact.

* Apply this proposition to prove directly that the area of the triangle contained by the tangent and the asymptotes is constant.

7. * The section of a right cone by a plane parallel to a line in its surface, and perpendicular to the plane containing that line and the axis, is a parabola.

GEOMETRY OF TWO DIMENSIONS.

1. FIND the equation of the straight line which meets 1845
the axes of x and y respectively at distances a and b from the origin.

2. Investigate the polar equation of the conic sections referred to the focus.

If S be the focus, and PSP' any focal chord, then $\frac{1}{SP} + \frac{1}{SP'}$ = a constant quantity.

3. If the distance between the vertex and focus of an ellipse remain constant while the major axis increases *sine limite*, the ellipse will ultimately pass into a parabola.

4. Determine the equation of the diameter of a given system of parallel chords in an ellipse; and prove that if one diameter be parallel to the chords of another, the latter diameter will also be parallel to the chords of the former.

5. Find the locus of a point, from which if a pair of tangents be drawn to an ellipse, the straight line joining the points of contact shall pass through the focus.

6. Determine the nature of any plane section of a right cone.

7. Investigate the position of a point whose distance from any point of the curve $y^2 = lx + nx^2$ is a rational function of the abscissa.

8. A straight line and a curve of the n^{th} order cannot have more than n points in common.

9. Prove that the curve whose equation is $y + x \log_e x = 0$ touches the axis of y at the origin, but for a sufficiently small positive value of y lies further from it than any curve whose equation is $y = ax^n$ which touches it.

1. Find the angle contained between two straight lines, the 1846
equations of which are given.

2. Shew that the tangents at the extremities of any focal chord of a parabola intersect at right angles in the directrix.

3. CP, CD being semi-conjugate diameters of an ellipse, find the co-ordinates of D in terms of those of P.

4. Prove that in the ellipse $PF \cdot CD = AC \cdot BC$.

5. Find the general equation to a hyperbola referred to oblique axes parallel to its asymptotes, assuming its equation referred to its principal axes.

6. In any conic section, if two chords intersect, the ratio of the rectangle under the segments of the one to the rectangle under the segments of the other depends only on the directions of the chords.

7. Prove that an oblique cone on a circular base can be cut by a plane not parallel to the base so that the section shall be a circle.

Prove that the cone so cut off is similar to the whole cone.

8. If an equation of the second degree, $f(x, y) = 0$, represents two straight lines, then either the equations

$$\frac{df(x, y)}{dx} = 0, \quad \frac{df(x, y)}{dy} = 0,$$

are equivalent to but one equation, or else the values of x and y found from them considered as simultaneous satisfy the equation $f(x, y) = 0$.

Is any other condition necessary in order that the equation $f(x, y) = 0$ may represent two straight lines?

1847 1. Find the length of the perpendicular let fall from the point (a, b) upon the straight line whose equation referred to rectangular axes is $x \cos \alpha + y \sin \alpha = c$.

2. The tangents at the extremities of any chord of a parabola intersect in the diameter which bisects the chord.

3. Find the locus of a point the sum of whose distances from two fixed points is constant.

4. Find the equation to the parabola referred to the two tangents which meet in the point of intersection of the directrix and axis.

5. If from the several points of any straight line, the equa-

tion to which is given, pairs of tangents be drawn to an ellipse, prove that the corresponding chords of contact will all pass through a fixed point, and determine the co-ordinates of that point.

What does this proposition become when the given line passes through the centre of the ellipse?

6. Find the equation to the tangent at any point of an ellipse, in terms of the angle which it makes with the major axis.

7. Prove that the semi-conjugate axis of a hyperbolic section of a right cone is a mean proportional between the perpendiculars dropped from the vertices of the hyperbola upon the axis of the cone.

8. If $ay^2 + bxy + cx^2 + dy + ex + f = 0$,
$a'y'^2 + b'x'y' + c'x'^2 + d'y' + e'x' + f' = 0$,
be the equations to the same curve referred to any two systems of oblique co-ordinates, prove by means of the transformation of co-ordinates that $4ac > = < b^2$ according as $4a'c' > = < b'^2$.

1. Explain the method of representing a plane curve by means of an equation $f(x, y) = 0$, and shew that when this equation is of the first degree it represents a straight line. 1848

2. Determine the perpendicular distance of a given point from a given straight line.

3. Obtain the polar equation to the straight line.

4. If $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_4 = 0$ be the equations to the sides of a quadrilateral, shew that the equation $u_1u_3 + u_2u_4 = 0$ will be that to a circumscribing conic section.

1. Find the distance between two points in the same plane in terms of their co-ordinates, the axes being oblique. 1849

2. Find the polar equation of a conic section, the focus being pole; and the polar equation of the tangent at any point.

3. Determine the locus of the middle points of the chords of the curve

$$ax^2 + 2bxy + cy^2 + 2ex + 2fy + g = 0$$

which are parallel to the line

$$x \sin\theta - y \cos\theta = 0;$$

and hence find the position of the principal axes of the curve.

4. Determine under what circumstances the locus represented by the equation

$$ax^2 + 2bxy + cy^2 + 2ex + 2fy + g = 0,$$

is two parallel straight lines.

DIFFERENTIAL CALCULUS.

1845 1. If u be a function of the variables $y, z, \ldots$ each of which is a function of x, prove that

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} + \frac{du}{dz}\frac{dz}{dx} + \ldots$$

Apply this theorem to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from the equation $f(x, y) = 0$.

2. Define the circle of curvature, and find expressions for the radius of curvature and the co-ordinates of the centre of curvature at any point of a plane curve referred to rectangular co-ordinates.

3. Find the differential co-efficient of the volume of a solid of revolution.

1846 1. Shew how to find the maxima and minima of a function of one independent variable.

2. If $f(x)$ and its first n differential co-efficients do not become infinite from $x = a$ to $x = a + h$, prove that the error committed in taking instead of $f(a + h)$ the first n terms of its expansion by Taylor's theorem is equal to the value of $\frac{h^n}{\lfloor n} f^{(n)}(a + \theta h)$ for some value of θ between 0 and 1.

3. Shew that $-\frac{d\theta}{du}$ is the polar subtangent of a curve, u being the reciprocal of the radius vector; and that there is generally a point of inflection where $u + \frac{d^2u}{d\theta^2} = 0$.

4. At points of a plane curve for which the curvature is a maximum or minimum, the circle of curvature has a contact of a higher order than the second.

5. Shew how to find the equation to a curve touching all the curves included in a given equation involving n parameters, there being $n - 1$ equations of condition among the parameters.

1. Find the expression for the radius of curvature in terms of p and r. 1847

2. Shew how the differential calculus may be applied to determine the limiting value of a fraction $\frac{f(x)}{\phi(x)}$ which, for a particular value of x, assumes the form $\frac{0}{0}$. Prove that the same rule applies if the fraction assume the form $\frac{\infty}{\infty}$.

3. If r and θ be the polar co-ordinates of any point of a curve, shew that at a point of inflexion

$$1 + \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 = \frac{d}{d\theta}\left(\frac{1}{r}\frac{dr}{d\theta}\right).$$

Why does this formula fail to determine the point of inflexion when the tangent at that point passes through the pole?

4. Shew how to find the envelope of the system of curves represented by the equation $F(x, y, a) = 0$, proving that it touches all the curves of the system.

5. If u be a homogeneous function of n dimensions of the independent variables, $x, y, z\ldots$, prove that

$$n(n-1)\ldots(n-r+1)\,u = \left(x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz}\ldots\right)^r u,$$

where the signs of differentiation are supposed to affect u only.

1. If $u = 0$ be the equation to a plane curve, prove that the equation to its tangent at the point (x, y) is 1848

$$(x' - x)\frac{du}{dx} + (y' - y)\frac{du}{dy} = 0.$$

How is the direction of the tangent ascertained at a point whose co-ordinates make

$$\frac{du}{dx} = 0 \text{ and } \frac{du}{dy} = 0?$$

2. If S be the surface generated by the revolution of an arc s of a plane curve about the axis of x prove that $\frac{dS}{ds} = 2\pi y$.

3. Determine the conditions in order that $f(x, y)$, a function of two independent variables, may have a maximum or minimum value for the values x_1 y_1 of x and y, and shew how to distinguish between the two cases.

4. If $F(x)$, $f(x)$ be two functions of x which vanish when $x = 0$, and which, as well as their differential co-efficients, vary continuously as x varies from 0 to any finite value, $x_1 f'(x)$ being supposed to retain the same sign during this variation, shew that $\frac{F(x_1)}{f(x_1)} = \frac{F'(\theta x_1)}{f'(\theta x_1)}$, θ lying between 0 and 1.

Shew how this theorem may be applied to determine the limiting value of a fraction which for a particular value of the variable assumes the form $\frac{0}{0}$.

1849 1. Define a differential co-efficient; and find that of x^n, without assuming the binomial theorem.

2. Define the terms maximum and minimum, and shew how to find when $f(x)$ is a maximum or minimum.

3. Shew how to find the locus of the points of ultimate intersection of a system of curves whose equation contains one arbitrary parameter; and prove that the locus in question touches each curve of the system at its point of intersection with the consecutive curve.

4. Show how to change the independent variable in a differential equation from x to a given function of x

5. If three branches of a curve whose equation is $u = 0$ pass through a point, u being a rational function of x and y, shew that the partial differential co-efficients of u of the first and second order all vanish for that point. Obtain an equation for determining the directions of the branches.

6. Find the equations of a trochoid, and determine its points of inflexion.

7. If $f(x)$ and $f'(x)$ be finite and continuous for all values of x, real or imaginary, of which the modulus ρ is less than R, prove that for values of ρ less than R the arithmetic mean of the n values of $f(\rho\epsilon^{t\sqrt{-1}})$ for values of t increasing by $\frac{2\pi}{n}$ approaches to $f(0)$ as its limit when n increases indefinitely.

Prove that for such values of x, $f(x)$ can be expanded in a convergent series, according to ascending integral powers of x.

INTEGRAL CALCULUS.

1. INVESTIGATE the integrals of 1845

$$\frac{dx}{\sqrt{x^2 - a^2}} \text{ and } \frac{dx}{x\sqrt{x^2 - a^2}}.$$

2. Investigate the relation between the amplitudes of two elliptic functions (having the same modulus) and that of a third equal to their sum or difference.

1. Integrate 1846

$$\frac{dx}{\sqrt{a^2 + x^2}}, \quad \frac{dx}{\sin x}, \quad (a^3 + x^3)^{-\frac{1}{3}}\, dx, \quad \frac{dx}{(a^2 + x^2)^2},$$

and find the value of $\int_0^{\frac{\pi}{2}} (\sin\theta)^n\, d\theta$, where n is a positive integer.

2. If $\frac{d\phi(x)}{dx} = f(x)$, and if $f(x)$ does not become infinite between $x = a$ and $x = b$, and if the interval $b - a$ is divided into n intervals $h_1, h_2 \ldots h_n$, prove that $\phi(b) - \phi(a)$ is the limit of the sum of the series

$$f(a)\, h_1 + f(a + h_1)\, h_2 \ldots + f(a + h_1 \ldots + h_{n-1})\, h_n$$

when the number of the intervals is indefinitely increased and their magnitudes are indefinitely diminished.

3. Find the volume of the solid generated by the revolution, about the axis of x, of the lemniscata the equation of which is $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$.

4. Prove that $\Gamma(m)\, \Gamma(1 - m) = \frac{\pi}{\sin m\pi}$.

1847 1. Integrate $\frac{1}{x^2 - a^2}$ $\frac{1}{x\sqrt{x^2 + a^2}}$, and $e^{ax} \sin mx$; and find the value of $\int_0^\infty x^n e^{-x}\, dx$, n being a positive integer.

2. Find the value of $\int_0^\infty \epsilon^{-x^2} dx$, and hence shew that

$$\int_0^\infty \frac{\sin x}{\sqrt{x}} \cdot dx = \int_0^\infty \frac{\cos x}{\sqrt{x}} \cdot dx = \sqrt{\frac{\pi}{2}}.$$

3. Shew how to find the equation to the evolute of a plane curve referred to rectangular axes.

1848 If $\frac{df(x,c)}{dc}$ be finite between the finite limits a and b of x, prove that $\frac{d}{dc}\int_a^b f(x,c)\, dx = \int_a^b \frac{df(x,c)}{dc} dx$. Why may the theorem fail if a, b, or $\frac{df(x,c)}{dc}$ be infinite?

1849 Integrate $\frac{dx}{\sqrt{a^2 - x^2}}$.

GEOMETRY OF THREE DIMENSIONS.

1845 1. Prove that the equation to a plane passing through the origin of co-ordinates is $z = Ax + By$. Assign a geometrical meaning to A and B when the axes are oblique.

From a given point O, OP is drawn meeting a given plane in Q, and the rectangle contained by OP, OQ is invariable; what is the locus of P?

2. Prove that in every surface of the second order which has a centre, there are three principal diametral planes at right angles to one another.

3. If ρ, ρ' be the two principal radii of curvature at any point of a curve surface, and R the radius of curvature of a normal section inclined at an angle α to the principal section corresponding to the former,

$$\frac{1}{R} = \frac{\cos^2 \alpha}{\rho} + \frac{\sin^2 \alpha}{\rho'}.$$

1846 1. Determine the condition of two lines in space referred to rectangular co-ordinates being at right angles to each other.

2. Define a line of curvature on any surface; and shew that the differential equation of its projection on the plane of xy is of the form

$$P\left(\frac{dy}{dx}\right)^2 + Q\frac{dy}{dx} + R = 0,$$

where P, Q, R are functions of x and y.

3. Find the differential equation to a conical surface; and thence find the equation to the conical envelope of the surface $ax^2 + by^2 + cz^2 = d$.

1. Find the length of the perpendicular dropped from a 1847
given point upon a given plane, and the projections of this perpendicular upon the co-ordinate axes.

2. Assuming the expression $R = \dfrac{1 + \tan^2\theta}{r + 2s\tan\theta + t\tan\theta}$ for the radius of curvature of a normal section of a curve surface, deduce Euler's theorem,

$$\frac{1}{R} = \frac{1}{\rho}\cos^2\theta + \frac{1}{\rho_1}\sin^2\theta.$$

What are the analytical circumstances under which this theorem fails? Give a geometrical illustration. Does Meunier's theorem fail in such cases?

3. If a surface be generated by a straight line, and the tangent plane at any point be also the tangent plane at any other point in the generating line which passes through the first point, prove that $rt - s^2 = 0$.

1. Define conjugate diameters of a surface of the second 1848
order, and prove that the sum of the squares of any system of conjugate diameters in an ellipsoid is the same.

2. If $z = f(x, y)$ be the equation to a surface, interpret the conditions obtained.

3. Find the equation to the tangent line and normal plane respectively at any point of a curve whose equations are $u = 0$, $v = 0$.

4. Find the general integral equation, and the partial differential equation to surfaces of revolution.

1849 1. Find the equation of the plane which contains the line whose equations are

$$Ax + By + Cz = 1,$$
$$A'x + B'y + C'z = 1,$$

and the point whose co-ordinates are $\alpha\beta\gamma$.

2. Express the area of a triangle in space, in terms of the rectangular co-ordinates of its angular points.

3. Find the relations among the coefficients of the general equation of the second order when it represents a surface of revolution.

4. Investigate the differential equations of the lines of curvature of an ellipsoid; and prove that the lines of curvature are the intersections of cones of the second degree with the surface of the ellipsoid.

DIFFERENTIAL EQUATIONS.

1845 1. INTEGRATE the partial differential equation

$$M\frac{dz}{dx} + N\frac{dz}{dy} = P,$$

M, N and P being functions of x, y and z.

Distinguish between the *complete primitive* and the *general primitive* of a partial differential equation of the first order.

2. If the complete integral of the linear equation

$$f\left(\frac{d}{dx}\right)y = 0 \ldots\ldots\ldots (1)$$

be known, shew how to integrate the equation

$$f\left(\frac{d}{dx}\right)y = \phi(x) \ldots\ldots (2);$$

and if the coefficients of the equation (1) are constant, and the roots of the equation $f(u) = 0$ are a_1 a_2 . . a_n, prove that the integral of (2) is

$$y = e^{a_1 x}\int dx\, e^{(a_2 - a_1)x}\int dx\, e^{(a_3 - a_2)x} \ldots \int dx\, e^{(a_n - a_{n-1})x}\int dx\, e^{-a_n x}\phi(x).$$

1846 Given the equation of a family of curves involving an arbitrary parameter, shew how to find the equation of the family of orthogonal trajectories.

Find the equation to a curve such that the tangent termi- 1847
nated by the axis of x is of a constant length, and prove that its evolute is a common catenary.

1. Shew how the variables may be separated in a homo- 1848
geneous differential equation; and find a factor which will render it integrable.

2. When P, Q, and R are functions of x, y, z if by any combinaton of the two equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ we can obtain two relations $M = A$, $N = B$ between x, y, z, where A and B are arbitrary constants, shew that $M = \phi(N)$ is the solution of the equation $P\frac{dz}{dx} + Q\frac{dz}{dy} = R$.

1. Shew how to determine whether a given solution of a 1849
differential equation of the first order is a singular solution or a particular integral.

2. Integrate the linear equation

$$f\left(\frac{d}{dx}\right)y = 0,$$

the coefficients being constant, and the roots of $f(z) = 0$ possible and unequal.

If $f(z)$ involve the factor $\{(z+\alpha)^2 + \beta^2\}^r$, find the corresponding part of the integral.

ELEMENTARY STATICS.

1. ASSUMING that when two forces are represented in mag- 1845
nitude and direction by two adjacent sides of a parallelogram the diagonal is in the direction of the resultant, prove that it also represents it in magnitude.

2. Find the relation between P and W on a smooth inclined plane, P acting in a given direction. In what direction must P act to support the greatest weight.

3. Find the centre of gravity of the frustum of a pyramid

bounded by a plane parallel to the base, assuming the position of that of an entire pyramid.

1846 1. What is meant by the *resolved part* of a force in any direction? Assuming the rule for the resolution of forces, determine the resultant of any number of forces acting at the same point.

How may the rule for the resolution of forces be deduced from the equilibrium of a chain on an inclined plane?

2. Find the relation of the power to the weight when there is equilibrium on the screw, friction not being considered.

Explain the principle of Hunter's screw.

3. Define the *centre of gravity* of a material system; and find the centre of gravity of two particles P and P'.

If r be the distance from P of a point in the line joining P and P', and r' its distance from P' shew that $Pr^2 + P'r'^2$ is least when that point is the centre of gravity.

1847 1. Assuming the parallelogram of forces for the direction, prove it for the magnitude of the resultant.

2. Find the relation between the power and the weight on the inclined plane, (1) when the power acts parallel to the plane, (2) when it acts horizontally.

3. Find the centre of gravity of a triangle.

1848 1. If two forces acting on a particle, be represented by two adjacent sides of a parallelogram, prove that their resultant will act in the direction of the corresponding diagonal, pointing out any assumptions or propositions which you may employ in the proof.

2. * When a weight is supported on a smooth inclined plane by a force along the plane, the force is to the weight as the height of the plane is to its length.

3. * Find the relation of P to W in the system of pulleys where each string is attached to the weight; and prove that $Pp = Ww$, where p, w, are the spaces gone through by P and W respectively when the system is put in motion.

4. * When a body is kept in equilibrium by three forces acting in one plane, either their directions are parallel, and one force is equal to the sum or difference of the other two, or their directions meet in a point, and each force is as the sine of the angle between the other two.

5. * Assuming the principle of the straight lever for two forces, find the condition of the equilibrium of a rigid body moveable about a fixed axis, and acted on by any number of forces in a plane perpendicular to the axis.

6. * Prove that the statical effect due to the weights of the several particles of which a body is composed is the same as it would be if all the matter were collected at its centre of gravity.

1. * Assuming the parallelogram of forces, so far as the direction of the resultant is concerned, shew that the diagonal of the parallelogram represents the magnitude of the resultant. 1849

2. * Find the relation between P and W in the system of pullies in which the same string passes round all the pullies.

3. * State and explain the conditions necessary and sufficient for the equilibrium of a body which has one or more points in contact with a smooth plane, and is acted upon by any forces.

4. * Explain the nature of the action and reaction of smooth surfaces in contact.

5. * Find the relation between the power and the weight upon the screw; and shew that

$$\frac{P}{W} = \frac{W\text{'s virtual velocity}}{P\text{'s virtual velocity}}.$$

6. * Find the centre of gravity of a triangle.

7. Explain how the general equation of virtual velocities is applied in Statics, and state the kind of problems in which it is most convenient. How is it to be applied to determine any required tension or reaction?

ANALYTICAL STATICS.

1845 1. DETERMINE the conditions of equilibrium of a rigid body acted on by any system of forces, one point in the body being fixed.

2. The area of the surface generated by a plane curve which turns about an axis in its own plane is equal to the product of the length of the curve into the length of the path of its centre of gravity, provided the generating curve lies wholly on one side of the axis. Why is this last condition necessary?

1846 1. Define a *couple*, and shew that no single force can counteract its effect.

2. When a system of forces acting upon a rigid body is reducible to a single resultant, find the equations to the line in which it acts; and thence deduce the conditions of equilibrium, by considering the direction of the resultant in this case as indeterminate.

3. Find the tension at any point of a common catenary.

1847 1. Prove the parallelogram of couples, shewing how the directions in which the couples act in their own planes are taken into account.

2. Shew that the surface generated by the revolution of a plane curve about an axis in its plane is equal to the rectangle of which the sides are the length of the curve and the length of the path of its centre of gravity.

How must the enunciation be altered if the curve lie on both sides of the axis?

1848 1. Prove that a system of forces acting on a rigid body is always reducible to a single force and a single couple in an infinite variety of ways.

Shew that however this reduction is effected the force is always the same in magnitude and direction, but the couple may be different, and has its least moment when its axis is parallel to the direction of the force.

2. A figure of revolution rests in equilibrium upon its vertex on a rough horizontal plane; shew that the stability or instability of the equilibrium depends upon the centre of curvature of the generating curve at the vertex being above or below the centre of gravity of the solid.

1. Prove that the effect of a couple is not altered by turning 1849
its arm through any angle in its own plane.

2. Find the equation to the common catenary.

3. A fine inextensible string passes over a plane rough curve, and is acted upon at its extremities by two forces; find the ratio between the forces when the string is on the point of slipping.

ELEMENTARY DYNAMICS.

1. How are velocity and accelerating force measured? 1845
State the second law of motion.

2. A perfectly elastic ball impinges directly upon another at rest; determine the velocities after impact.

3. Two unequal weights connected by a string hang over a fixed pulley. Find the accelerating force, neglecting the inertia of the pulley.

4. Find the equation to the path of a projectile in vacuo.

1. How is accelerating force measured? Prove that the 1846
space described from rest by a particle under the action of a uniform accelerating force varies as the square of the time.

2. Shew that the velocity of a projectile, at any point of its path in vacuum, is equal to the velocity which would be acquired in falling freely from the directrix of the parabolic path to that point.

1. Explain the two modes of measuring force in dynamics. 1847
What is the ratio of the accelerating forces when equal pressure act on different bodies?

2. A body is projected vertically downwards with a given

velocity; shew that the space described in any time is equal to that which would be described in the same time with a uniform velocity equal to half the sum of the velocities at the beginning and end of the time. Also find the velocity of the body in terms of the space described.

3. Prove that a body projected in any direction inclined to the vertical, and acted on by gravity, will describe a parabola, and shew that the velocity at any point is that which would be acquired in falling from the directrix.

1848 1. * Enunciate the first and second laws of motion, and mention experimental facts which would lead to their assumption.

* What is the nature of the final evidence which is considered conclusive as to the truth of these laws?

2. * Shew that in uniformly accelerated motion $s = \frac{1}{2}ft^2$, proving, if your method require it, but not assuming, that if the velocity of the body be reversed the backward motion will be exactly similar to the forward motion.

3. * The time of descent down any chord passing through the highest or lowest point of a vertical circle is the same as the time down the vertical diameter.

4. Prove that a body projected obliquely and acted on by gravity will describe a parabola.

* Find the velocity and direction of projection, in order that the projectile may pass horizontally through a given point.

5. State and explain the third law of motion.

* Can we form a conception of mass without introducing the idea of weight?

6. * A body, whose mass is m, is projected with a velocity V, and acted on by a constant pressure P in the line of projection, find the velocity of the body at any time; and if the pressure act in a direction opposite to that of projection, find how long it will be before the body is brought to rest.

7. * Find the accelerating force when one weight pulls

another over a fixed pulley. Find also the tension of the string.

8. * If two imperfectly elastic balls moving with given velocities in a straight line impinge directly, find their velocities after impact.

9. * Find the time of oscillation of a body oscillating in a cycloid.

1. * State the second law of motion and explain its use; and mention some experiments which give results in accordance with it. 1849

2. * Distinguish between *accelerating* and *moving* force. Explain what is meant by the phrase "action and reaction are equal and opposite."

3. * If a body be projected with a velocity u, and acted on by a uniform force f in the direction of motion, shew that the space passed over in the time t will be $ut + \frac{1}{2}ft^2$

4. * Two equal balls A and B are moving with given velocities in the same plane in directions at right angles to each other, and the line joining their centres at the instant of impact is in the direction of A's motion. Determine their motions after impact, supposing them smooth and inelastic.

5. A heavy particle is drawn up an inclined plane by means of another attached to it by a string passing over the upper edge of the plane, the latter particle descending vertically; find the accelerating force.

* Find also the tension of the string.

6. * The velocity of a projectile at any point of its parabolic path is that which would be acquired by a body falling freely from the directrix to that point.

7. * A heavy particle slides down an inclined plane of given height under the action of gravity; find the time of descent and the velocity acquired.

8. * How may a pendulum be made to oscillate in a cycloid?

NEWTON. 1, 2, 3.

1845 1. ENUNCIATE and prove Newton's tenth Lemma. . Adapt the figure to the case in which the force is constant; and deduce the formula $s = \frac{1}{2} f t^2$.

2. When bodies describe different circles with uniform velocities, the forces tend to the centres of the circles, and are as the squares of the velocities divided by the radii of the circles.

1846 1. Enunciate and prove Newton's fourth Lemma.

2. If a body moves in any orbit about a fixed centre of force, the areas described by lines drawn from the centre to the body lie in one plane, and are proportional to the times of describing them. NEWTON, *Princip.* Lib. I. Prob. 1.

3. A body revolves in an ellipse; find the law of force tending to one of the foci. NEWTON, *Princip.* Lib. I. Prop. 11.

1847 1. State and prove Newton, Lemma X. How will the figure and the result be modified, if the force be a uniform one?

2. If a body move in any orbit about a fixed centre of force, the areas described by lines drawn from the centre to the body lie in one plane, and are proportional to the times of describing them. Prove this, and shew that the reasoning applies whether the force be attractive or repulsive.

3. Find the velocity at any point of an ellipse described about a centre of force in the focus.

1848 1. * Enunciate and prove Newton's first Lemma. In what Lemma is the idea of the ultimate ratio of two vanishing quantities first introduced?

Give Newton's answer to the objection, "that there can be no ultimate ratio of vanishing quantities, since their ratio cannot be considered ultimate before they have vanished, and after they have vanished there can be no ratio at all."

2. * State and prove Lemma X, and shew that under the circumstances of the enunciation, force = 2 limit $\frac{\text{space}}{(\text{time})^2}$.

3. * If a body revolve about a fixed centre of force, the areas described by lines drawn from the body to the centre of force lie in one plane, and are proportional to the times of describing them.

Point out the laws of motion assumed in the proof of this proposition.

4. * The centripetal forces of bodies which describe different circles with uniform velocities tend to the centres of the circles, and are to each other directly as the squares of the arcs described in the same time, and inversely as the radii of the circles.

5. * A body describes an ellipse; required the law of force tending to the centre.

Give the reasoning by which Newton extends the result obtained to the case of a parabola, or an hyperbola. Is the same kind of reasoning applicable when the centre of force is at the focus, instead of at the centre?

6. * Find the law of force tending to the focus of a parabola.

7. * State Kepler's laws; and enunciate the various propositions in Newton by means of which they may be deduced from the theory of universal gravitation.

How does this theorem account for the deviations from the exactness of these laws which accurate observations have shewn to exist?

8. Define the radius of curvature at any point of a curve; and shew that in a conic section the radius of curvature

$$= \frac{(\text{normal})^3}{(\frac{1}{2} \text{ latus rectum})^2}.$$

9. State and prove Newton's Lemma VII.

1. * Enunciate and prove Newton's fourth Lemma. 1849

2. * Define similar curves, and shew that all parabolas are similar to each other.

3. * Enunciate and prove Lemma IX.

4. * A body which moves in any plane curve, and a radius being drawn to a point either fixed or moving uniformly in a right line, describes areas about that point proportional to the times, is acted on by a centripetal force tending to that point.

5. * A body describes a parabola under the action of a force parallel to the axis; determine the law of force.

Find the velocity at any point, and the time of moving from the vertex to the extremity of the latus rectum.

6. * Find the law of force under the action of which a body may describe an ellipse, one of the foci being the centre of force.

If v, v' be the velocities at the extremities of any focal chord, and u that at the extremity of the latus rectum, then will v^2, u^2, v'^2 be in arithmetical progression.

7. * A body moves in the circumference of a circle; find the law of force tending to an external point. Is the force attractive or repulsive?

8. * Given the velocity at any points whatever of an orbit described by a body under the action of a central force, find the position of the centre.

DYNAMICS PARTICLE.

1845 1. PROVE that in central orbits the velocity varies inversely as the perpendicular on the tangent. And hence shew that the general equation of central orbits is

$$h^2\left(\frac{1}{p^2}-\frac{1}{p_1^2}\right)=2\int_r^{r_1} P\,dr,$$

where P is the force at distance r, and r_1, p_1, are any two co-ordinate values of r and p.

2. When a particle moves under the action of any forces in one plane, the effective accelerating force on the particle at the time t is equivalent to $\frac{dv}{dt}$ in the direction of the tangent, and $\frac{v^2}{\rho}$ in that of the normal to the path described, v being the velocity at the time t and ρ the radius of curvature.

Apply the above expressions to find the orbit described by a particle about a centre of force varying directly as the distance, and find where the tangential force is greatest.

3. Investigate formulæ for determining the effect of a small disturbing force on the oscillations of a simple cycloidal pendulum.

Hence shew that the time of oscillation will not be sensibly affected by the resistance of the air, supposing the amount of the resistance to depend solely on the velocity of the pendulum.

4. A particle moves on a smooth surface, and is acted on by no force except the reaction of the surface; determine the differential equation of its path.

1. Shew that when a body moves in an orbit round a 1846
centre of force, the velocity at any point of the orbit is that which would be acquired by the body in falling through a space equal to one-fourth of the chord of curvature, under the action of the force continued constant.

2. Prove that in a central orbit, the force being any function of the distance, there cannot be more than two apsidal distances.

3. Find the velocity of a body which describes an ellipse about a centre of force in the focus.

4. A particle being constrained to move on a curve, find the pressure, (1) when the curve is plane, (2) when of double curvature.

5. A body acted on by gravity is projected in a given direction in a medium of which the resistance varies as the

velocity; find the equation to the path described, and prove that it has a vertical asymptote.

1847 1. Find the differential equation between u and θ to the path of a body under the action of a central force; and shew how the constants which occur in its complete integral may be determined, when the velocity, distance, and direction of projection are given.

2. When a heavy particle moves on a curve in a vertical plane, find the pressure upon any point of the curve.

3. State Kepler's Laws. What information does each of them afford respecting the force which retains the planets in their orbits?

4. Find the time of falling to the centre when the force varies as the distance. If a particle be placed at rest at the centre of force, and the centre then begin to move uniformly in a straight line, find the time in which the particle will return to the centre.

5. When a particle is in motion under the action of any forces, shew that the osculating plane at any point passes through the direction of the resultant force at that point.

6. A particle in motion is acted on by any number of forces tending to fixed centres and varying as any functions of the distances; prove that the velocity of the particle at any instant depends only on its initial velocity and position, and on its position at the instant considered.

What step of the reasoning by which this proposition is established fails if the centres of force are in motion?

7. If r and θ be the polar co-ordinates at the time t of a particle moving in a plane under the action of any forces, shew that the forces acting on the particle are equivalent to $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$ in the radius vector, and $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$ perpendicular to the radius vector.

1. From the second law of motion deduce the equations of motion of a particle referred to three axes, rectangular or oblique. 1849

2. Find the velocity acquired by a particle in falling from an infinite distance to a given distance from a centre of force which varies inversely as the square of the distance. Find also the time employed in falling from a given distance to the centre. How may this time be deduced from the periodic time in an elliptic orbit?

3. Explain what is meant by centrifugal force, and shew how by its introduction mechanical problems relating to uniform rotation may be reduced to statical problems.

4. A body is acted on by a repulsive force varying inversely as the distance from a fixed centre, determine the motion, supposing it rectilinear.

5. When a particle moves under the action of a central force P investigate the differential equation to its orbit.

$$\left(\frac{d^2u}{d\theta^2} + u\right) h^2u^2 = P.$$

If two particles move about one another under their mutual attractions, shew how the computation of the motion of either relatively to the other is reducible to the case of motion about a fixed centre of force.

6. Determine the motion of a material particle which falls from rest under the action of a constant force, through a medium the resistance of which varies as the square of the velocity. Shew that the velocity of the particle never exceeds a certain limiting value, to which it approximates as the time increases.

DYNAMICS. SYSTEM.

1. THE motion of rotation of a rigid system acted on by any forces, about its centre of gravity, is the same as if the centre of gravity were fixed, and the same forces acted. 1845

2. Apply D'Alembert's principle to prove the general dynamical equation

$$\Sigma m \left\{ \left(X - \frac{d^2x}{dt^2} \right) \delta x + \left(Y - \frac{d^2y}{dt^2} \right) \delta y + \left(Z - \frac{d^2z}{dt^2} \right) \delta z \right\} = 0,$$

and hence deduce the six equations of motion of a rigid body.

3. Prove that in a compound pendulum the centres of oscillation and suspension are reciprocal.

4. Find the position of the plane, with respect to which the sum of the moments of the momenta of the different particles of a material system is a maximum, none of the forces of the system being extraneous to it, and shew that this position is invariable.

In the application of this result to the solar system, is it necessary to take into account the motion of the sun and planets about their respective axes?

1846 1. Express the moment of inertia of a body about any axis passing through a given point in terms of the moments about the principal axes through that point.

2. Enunciate and prove the principle of Least Action.

3. Explain the principle of the Gridiron Pendulum.

4. Prove the principle of *Vis Viva*.

5. Obtain equations for the determination of the motion of a rigid body which, being previously in motion, is acted upon by given impulsive forces.

1847 1. Apply D'Alembert's principle to prove that the angular motion of a rigid body oscillating about a horizontal axis is exactly the same as that of a simple pendulum of a certain length.

2. Prove that any possible instantaneous motion of a rigid body, one point of which is fixed, is equivalent to a motion of rotation about an axis passing through that point. Explain what has been termed the angular velocity of the body about any other axis through the fixed point, and shew that this is not the angular velocity of every particle of the body about that axis.

3. A body at rest is struck by a given impulsive couple, find the axis about which it begins to revolve, and the initial angular velocity.

1. State and prove the principle of the Invariable Plane, as relating to the motion of a material system relative to its centre of gravity. 1848

2. In any system of rigid bodies, prove that the increase of *vis viva* in passing from one position to another is measured by twice the whole labouring force expended, whether arising from external or from internal forces.

If the external forces consist of attractions directed to fixed centres of force, and of the reactions of smooth fixed surfaces, &c., and the internal forces consist of mutual attractions, and of the reactions of smooth surfaces, &c., prove that the labouring force expended in any time depends only on the positions of the particles at the beginning and end of that time, the attractions being supposed to vary as any functions of the distances.

Does this proposition cease to be true if in the course of the motion any of the bodies or particles quit the surfaces on which they had previously been moving?

3. Assuming that the most general instantaneous motion of a rigid body moveable about a fixed point is a motion of rotation, prove that the most general instantaneous motion of a free rigid body consists of a motion of rotation about a certain axis combined with a motion of translation along the axis.

4. Assuming the general equations in x, y, z for the rotation of a body about its centre of gravity, obtain the corresponding equations in $\omega_1, \omega_2, \omega_3$, the angular velocities about the principal axes.

1. Explain how the reciprocity of the centres of suspension and oscillation has been employed in determining the length of the seconds' pendulum. What other methods have been employed for that purpose? 1849

If the observations are made in air, what corrections are required to reduce them to a vacuum?

2. Apply the principle of *vis viva* to determine under what conditions the equilibrium of a material system is stable.

3. When a rigid body, acted upon by no forces, is in motion about a fixed point, the resolved part of the angular velocity about an axis normal to the invariable plane is constant throughout the motion.

Describe a geometrical construction by means of which the instantaneous velocity and the axis about which it takes place may be represented at any instant.

HYDROSTATICS.

1845 1. FIND the conditious of equilibrium of a solid floating in a fluid.

2. Shew how to compare the specific gravities of two liquids by weighing the same solid in each.

3. Explain the action of the syphon.

Water is flowing out of a vessel through a syphon; what would take place if the pressure of the atmosphere were removed and afterwards restored, (1) when the lower end of the syphon is immersed in water, (2) when it is not?

4. Investigate a formula for determining by means of a barometer, the altitude of a balloon.

1846 1. Find the resultant of the pressure of a heavy fluid on the surface of a solid wholly or partially immersed in it.

2. What is the *dew point?* State accurately what we determine by the observation of it with reference to the state of the air.

3. Define the *centre of pressure* in the case of any curve surface submitted to fluid pressure. Is their necessarily such a point?

4. Find the tension at any point of a flexible cylindrical vessel containing fluid.

5. Shew that at all points in a surface of equal pressure in a fluid at rest, the resultant of the forces is perpendicular to the surface, and that when the fluid is elastic and of the same kind

throughout, surfaces of equal pressure and density are also surfaces of equal temperature.

6. What is the precise definition of a *degree of temperature* in any thermometer? Why is it necessary to observe the barometer in graduating a thermometer?

1. Find the conditions which must be satisfied when a solid floats at rest in a fluid. 1847

2. Explain how a thermometer is graduated, and shew how to compare the scales of two differently graduated thermometers.

3. A hollow cylinder closed at both ends is partly filled with a homogeneous liquid, not acted on by gravity, which revolves uniformly about the axis of the cylinder; find the force which tends to separate the two portions into which the cylinder is divided by an imaginary plane passing through its axis.

4. Define the Metacentre; and show how its position enables us to determine the nature of the equilibrium of a floating body.

1. *Define a fluid, and prove that a pressure applied to the surface of a fluid mass in equilibrium will be equally transmitted in all directions. 1848

2. In a mass of homogeneous liquid in equilibrium under the action of gravity, the difference of pressure at two points in the same vertical line which falls wholly within the fluid is proportional to the distance between those points.

*What further considerations are necessary to prove this proposition when the straight line joining the two points in question does not fall wholly within the fluid.

3. *When a liquid is in equilibrium in a vessel, prove that the vertical component of the pressure on any portion of the surface beneath is equal to the weight of the superincumbent column of the fluid, supposing the column to reach to the surface without interruption.

How must the enunciation be altered so as to make the proposition apply (1) to the case in which the column is interrupted; (2) to the case in which the surface considered is pressed upwards?

4. * When a solid floats in a liquid, its weight is equal to that of the fluid displaced, and its centre of gravity is in the same vertical line with that of the fluid displaced.

Is this statement true in a case of a body partly immersed in several liquids which are superposed?

5. * Describe the experiment which shows that the pressure of air is proportional to its density while the temperature remains constant. State the relation between the pressure, density, and temperature of air.

6. * Describe the nature and use of the diving-bell.

7. * Describe the siphon and its action.

8. * Define the specific gravity of a substance; and shew how the specific gravity of a solid may be found by weighing it in air and water. How is the method modified if the solid floats in water?

9. * Describe the construction and action of the condensing steam engine. In what respect was this an improvement upon the atmospheric engine?

10. Investigate the equations of fluid equilibrium.

$$\frac{dp}{dx} = \rho X, \frac{dp}{dy} = \rho Y, \frac{dp}{dz} = \rho Z.$$

Shew that if $Xdx + Ydy + Zdz$ be a perfect differential, the fluid mass will be arranged in surfaces of equal pressure and density, of which no two intersect. If forces existed which did not render the above expression a perfect differential, and a fluid mass were in equilibrium under the action of such forces, shew that the surfaces of equal pressure could not be also surfaces of equal density.

11. A straight tube closed at one end, and open at the other, revolves with a uniform angular velocity about an axis

meeting the tube at right angles; neglecting the action of gravity, determine the law of density of the air within the tube, and compare the pressure at the closed end with the atmospheric pressure.

1. *Distinguish between compressible and incompressible fluids. Explain how fluid pressure is measured. 1849

In the equation $p = k\rho$, which connects the pressure and density of an elastic fluid, k is a quantity of two dimensions with respect to space; shew why k is a quantity of this kind.

2. *The surface of a heavy incompressible fluid at rest is a horizontal plane.

Is this true practically of the surface of a liquid contained in a vessel of finite dimensions?

3. *Define specific gravity.

4. *Determine the whole pressure on a surface immersed in a heavy fluid of uniform density.

5. *Describe Nicholson's Hydrometer, and shew how it may be applied to compare the specific gravities of two fluids.

6. *If the atmosphere be supposod to be divided into indefinitely thin strata of equal thickness, the density of the air in those strata will be in geometrical progression.

7. *Describe Smeaton's air-pump, and find the density of the air in the receiver after any number of ascents of the piston.

8. *Determine the conditions of equilibrium of a floating body.

9. *Explain the formation of dew. Why is dew so much more copious in hot than in cold weather; and why is the appearance of abundance of dew in the morning an indication that the day will be fine?

10. Find the metacentre of a floating body which is symmetrical with regard to a vertical plane parallel to which the displacement is made.

11. What is meant by specific heat? If ρ' be the density of air under a given pressure, and $\mu\left(1-\frac{\rho}{\rho'}\right)$ be the increase of temperature due to a sudden change of density from ρ to ρ', (ρ' being not very different from ρ,) and if θ be the temperature corresponding to the density ρ, shew that

$$\frac{\text{specific heat of air under a given pressure}}{\text{specific heat of air under a given volume}} = 1 + \frac{\mu a}{1 + a\theta},$$

where a is the expansion of air for one degree of temperature.

GEOMETRICAL OPTICS.

1845 1. DEFINE the principal focus of a spherical reflector. In direct reflection at a spherical surface, the rectangle contained by the distances of the conjugate foci from the principal focus is invariable.

2. Find the deviation of the axis of a pencil refracted through a prism in a plane perpendicular to the edge.

3. Prove the proposition in Optics on which the construction of Hadley's sextant depends. How do you find the index error?

4. Shew that the rays of a small pencil of light after oblique reflection at a spherical surface converge to, or diverge from, two focal lines.

5. A pencil of white light is incident directly on a thin convex lens of small aperture. Determine the position and radius of the least circle of chromatic aberration.

6. Describe Newton's telescope with Ramsden's eye-piece, and trace a pencil from a given point in the object to the eye.

7. Two convex lenses of the same substance have the same axis; determine the distance between them in order that the combination may form an achromatic eye-piece for an astronomical telescope.

1. A small pencil of light is incident obliquely upon a plane refracting surface; find the distances of the focal lines from the point of incidence of the axis. 1846

2. Investigate the position of the geometrical focus of a small pencil of rays reflected directly at a spherical mirror.

How would you deduce from this case the formula for a spheroidal mirror? Could you in like manner deduce the formula for the aberration of a pencil of rays not indefinitely small from the formula proved for the sphere?

3. How would you find practically the refractive index of a transparent substance? Explain carefully how you would determine experimentally the quantities which occur in any formula you may use in this process.

4. Explain the nature of the two kinds of achromatism exemplified in the object-glass and the eye-piece of a telescope respectively, shewing distinctly why the pencils which emerge from the eye-glass produce no sensation of colour.

5. Explain the manner in which the image of an object is formed upon the retina.

6. If a telescope be directed to a distant luminous point, the magnifying power will be measured by the breadth of the object-glass divided by the breadth of the emergent pencil.

7. Describe the compound refracting microscope; and explain the mode of estimating its magnifying power.

1. Find the deviation of a ray after two successive reflections at plane mirrors inclined to each other, the course of the ray lying in a plane perpendicular to their line of intersection. What must be the first angle of incidence, that at a third reflection the course of the ray may be exactly reversed? 1847

2. Find the relative index of refraction between two media, having given the refractive index between each of them and a third. Does the demonstration hold for all angles of incidence and refraction at the common surface of the two media?

3. Describe the simple astronomical telescope; and shew

that its magnifying power is measured, (1) by the ratio of the focal length of the object-glass to that of the eye glass, (2) by the ratio of the diameter of the object glass to that of the emergent pencil.

4. When a ray of light passes through a prism denser than the surrounding medium, in a plane perpendicular to the edge of the prism, the deviation is from the edge.

5. When is a single lens said to be equivalent to a system of lenses through which an excentrical pencil passes, the axis of the pencil before incidence being nearly parallel to the axis of the lenses? Find the single lens equivalent to two lenses separated by a given interval, and apply the result to Ramsden's eye-piece when used with an object-glass of great focal length.

6. Mention the construction of Huyghen's and Ramsden's eye-pieces, describing by figures the course of an excentrical pencil through each. What advantage does the latter possess, and why cannot it be made achromatic?

7. From the formula

$$\frac{1}{v_2} + \frac{1}{u} = \frac{2\cos\phi}{r}$$

for finding the secondary focus of a pencil of light reflected obliquely at a concave mirror, deduce the formula

$$\frac{1}{v} + \frac{1}{u} = \frac{2}{r} + \left(\frac{1}{r} - \frac{1}{u}\right)^2 \frac{y^2}{r}$$

for the aberration of a direct pencil.

1848 1. * Define a pencil of rays, converging rays, diverging rays, and the focus of a pencil of rays.

If diverging or converging rays be reflected at a plane surface, the foci of incident and reflected rays are on contrary sides of the reflector, and equally distant from it.

2. * Define conjugate foci, and shew that, in the case of reflection at a spherical surface, they lie on the same side of the principal focus, that they move in opposite directions, and meet at the centre and surface of the reflector.

3. * A ray of light cannot pass out of a denser into a rarer medium if the angle of incidence exceeds a certain limit.

4. * Parallel rays, refracted at a plane surface, continue parallel.

5. * Parallel rays, refracted at a convex spherical surface of a denser, or a concave of a rarer medium, into which they pass, are made to converge; and refracted at a concave spherical surface of a denser, or a convex of a rarer medium are made to diverge.

6. * If A, a be two points on the surfaces of a lens where the radii are parallel to each other, the incident and emergent parts of a ray of light which passes through the lens in the direction Aa will be parallel.

Define the centre of a lens, and shew that it is a fixed point.

7. * The image of a straight line formed by a plane refracting surface is a straight line.

8. * Describe the Astronomical Telescope; trace the course of a pencil of rays from any point of a distant object, and find the magnifying power.

9. Compute the places and lengths of the focal lines of a small pencil of parallel rays after oblique reflection at a convex spherical surface.

10. Explain the practical mode of determining the refractive index of a substance, for any fixed line of the spectrum, by measuring the minimum deviation of that line as seen through a prism. Shew also how to measure the refracting angle of the prism.

11. Find the magnifying power of the Astronomical telescope with Huyghen's eye-piece, the focal lengths of the lenses being given.

1. * Enunciate the laws of reflexion and refraction; and 1849
state what you consider the most searching test of the truth of the latter.

2. * When rays diverging from a point are incident on a plane mirror, prove that the reflected rays diverge accurately from a point.

3. * Find the geometrical focus of a pencil of parallel rays reflected at a spherical mirror.

4. * When diverging rays are incident nearly perpendicularly upon a spherical refracting surface, the distance of the focus of incident rays from the principal focus of rays coming in a contrary direction, is to its distance from the centre of the refractor, as its distance from the surface to its distance from the geometrical focus of refracted rays.

5. * Determine by a geometrical construction the principal focus of a lens of inconsiderable thickness.

6. * Describe the human eye, and the defects of long sight and short sight; and shew how they may be remedied by the use of spectacles.

7. * Draw a figure representing the course of an oblique pencil through Gregory's telescope, explaining the principal parts of the figure.

8. In a common astronomical telescope of the simplest form, how would the defects arising from the unequal refrangibility of different kinds of light shew themselves? Point out the corrections which are necessary to remove these defects.

9. A small pencil of light is incident on a plane refracting surface; find the positions of the primary and secondary focal lines.

PLANE ASTRONOMY.

1845 1. When the Sun has a given north declination, shew at what parts of the Earth he is visible, (1) during 24 hours, (2) during 12 hours.

2. Find the time at a given place from observing the altitude of a known star.

3. Explain the cause of aberration, and find the amount

and direction of the change it produces in the apparent place of a star.

4. State the ordinary methods for determining the latitude of the place of observation. In what manner may a transit instrument, whose axis is placed north and south, be used for this purpose?

5. If the polar axis of an equatoreal be slightly displaced, determine the corrections to be made in the readings off.

6. Describe the method of observing with the mural circle. How can incorrect graduation be detected in this instrument?

7. Shew how, by particular observations, the longitude of the node and the inclination of a planet's orbit may be determined.

8. Shew that

$$\sqrt{r}\,\sin\frac{v}{2} = \sqrt{a(1+e)}\,\sin\frac{u}{2},$$

$$\sqrt{r}\,\cos\frac{v}{2} = \sqrt{a(1-e)}\,\cos\frac{u}{2},$$

where u is the eccentric anomaly of a planet, v its true anomaly, r its radius vector, a the semi-axis major and e the eccentricity of its orbit.

9. Determine whether there are any places on the Earth's surface at which a given solar eclipse appears central, and, if they are, shew how to find their latitude and longitude.

1. Define the terms *vertical line, zenith, meridian:* shew 1846
how to draw a meridian line by means of the shadow of a vertical rod, neglecting the change of the Sun's declination.

2. In using Hadley's sextant at sea, shew how to correct for the height of the observer's eye above the level of the sea.

3. Explain the cause of astronomical refraction; and shew how to find the coefficient of refraction by observations on two circumpolar stars. Why would not one star be sufficient?

4. Explain the use of observing with the mural circle by reflection and direct vision at the same transit, and describe the manner of making the observations.

5. Shew how to find the parallax of a heavenly body by meridional observations made at two known places either in the same or different meridians.

6. Shew when the equation of time, supposing the Earth's orbit circular, is to be added to the apparent time, and find when it is the greatest.

7. Find a formula for the aberration of a star in declination.

Explain why the star γ Draconis was peculiarly favourable for Bradley's discovery of Aberration.

8. In a total eclipse of the Sun, shew how to find the latitude and longitude of those places on the Earth's surface at which the total darkness begins and ends.

9. Shew how to graduate a horizontal dial, and find the limits beyond which it is useless to graduate it.

1847 1. Explain the reason why navigators in sailing round the world gain or lose a day in their reckoning, according as they sail eastwards or westwards.

2. Explain the different modes of measuring time in use among astronomers. Find the mean time at which a known star crosses the meridian.

3. Find the length of the day at any given place. What is the lowest latitude at which the Sun does not set for 24 hours?

4. When the vertical plane in which a transit instrument moves nearly coincides with the meridian, find the deviation by observing the time of transit of two known stars.

What stars are the most eligible for the purpose?

5. Describe fully the method of determining the longitude by the distance of the Moon from the Sun, or from a fixed star.

6. What is meant by the aberration of light? In what plane does the aberration of a star take place?

7. Shew how to determine the declinations of the stars by observing their azimuths when near the horizon at an observatory situated near the equator, and point out the advantages of this method.

8. Shew how to determine the nadir point of a mural circle by observing the cross wires directly, and by reflection in a trough of mercury.

9. Shew how to determine the latitude and the time by observing the altitudes of two known stars.

10. Supposing the orbits of comets to be equally distributed through space, prove that their mean inclination to the plane of the ecliptic is the angle subtended by an arc equal to the radius.

11. Having given the ratio which the daily arc of retrogradation of a planet when in opposition bears to the Sun's daily motion, determine the distance of the planet from the Sun, supposing the orbits of the Earth and planet to be circular and in the same plane.

If the orbit of the planet be really an ellipse, find at what point of it the planet must be at the time of observation, in order that the distance found on the above hypothesis may coincide with the true distance.

1. * Describe the phenomenon of the Sun's apparent motion among the stars, and mention the two principal hypotheses which have been made to account for it. 1848

What are the chief arguments in favour of the received hypothesis, that the Earth moves round the Sun in an orbit inclined to the plane of the equator?

* When does the sun set at the point of the horizon opposite to that at which he rose?

2. * Explain the kinds of observations which a transit, a mural circle, and an equatoreal are respectively designed to make. Describe the process of finding the error and rate of a clock at an observatory.

3. * Distinguish between sidereal time, solar time, and mean solar time; and explain why the time indicated by the common clock is sometimes before, sometimes behind that indicated by the dial.

4. * Give an explanation of the phenomenon of refraction,

and point out the astronomical observations, made in the plane of the meridian, which are affected by it.

What is the cause of twilight; and why is its duration so much less in the tropics than in the higher latitudes?

5. * What is meant by the precession of the equinoxes; how is its existence manifested, and what is the physical cause of it?

6. * Account for the phases of the Moon in the course of a month, and shew how a lunar eclipse arises.

Why are the satellites of Jupiter more frequently eclipsed than the Moon?

7. * Explain how the finite velocity of light causes a difference between the real and apparent places of a fixed star. What additional consideration arises in the case of a planet?

8. * Shew how the Moon's motion among the stars is made to determine the longitude at sea.

9. Determine the Moon's parallax by observations made out of the plane of the meridian. How are the effects of refraction in the observations to be avoided?

10. Determine the longitude of the perihelion, neglecting its motion. How is the motion of the perihelion determined, and what correction is thence introduced in the preceding process?

11. Find the aberration of a given star in declination.

12. Shew how to find the error of level of a transit instrument by means of a spirit level, supposing the radii of the pivots equal.

How may an inequality in the radii of the pivots be detected, and its effect on the apparent error of level determined, by applying the spirit level directly, and after the transit instrument has been reversed?

13. When the vertical plane in which a transit instrument moves nearly coincides with the meridian, find the deviation by

means of the observed superior and inferior transits of the same circumpolar star.

How is it that an accurate result may be obtained, seeing that the difference of intervals which is the subject of observation is affected by the errors of level and collimation?

14. Determine the inclination of the ecliptic to the horizon at a given hour on a given day.

Trace the motion in the course of the day of its point of intersection with the horizon.

1. *Describe the change of appearance presented by the 1849
starry heavens in the course of a night. How would the phenomena be altered to a person travelling southwards; and how would the notion he might have previously formed that the earth was isolated in space be confirmed by his journey? How does it appear that the earth must be regarded as a point with respect to the distances of the stars?

2. *Describe the transit instrument, without entering into an account of its adjustments.

3. *Define the terms meridian, vertical line, zenith. How do we arrive at the conclusion, that the vertical line is not the line joining the place considered with the centre of the earth; and what are the physical causes why the two are different?

4. *How does it appear from observation that the apparent path of the sun among the fixed stars is a great circle? Why are the points of intersection of the ecliptic and equator called equinoxes?

Describe the diurnal and annual motions of the sun as they would appear to a person at the North Pole?

5. *Define sidereal time, solar time, and mean solar time.

How are differences of terrestrial longitude determined by means of chronometers?

6. *What is meant by the equation of time, and to what two causes is it principally due? How do the parts depending on these two causes respectively alter in the course of a year?

7. * Define parallax. By what sort of observations are the parallaxes of the Moon and Sun respectively determined? Why are transits of Venus more valuable for the latter purpose than transits of Mercury?

8. Describe the collimating eye-piece, and explain its uses.

9. Shew that the position of a microscope with regard to the pointer of the mural circle is of no consequence. What possible errors are avoided or diminished by using six microscopes placed at equal distances round the circumference of the circle?

10. Find the latitude by observing the Pole Star out of the meridian.

11. When is a planet said to be stationary? If the orbits be circular and in the plane of the ecliptic, shew that all planets will be stationary at some points in their orbits.

12. Shew how to graduate a horizontal dial.

NEWTON'S LUNAR THEORY.

1845 1. FIND an approximate expression for the tangential disturbing force of the Sun on the Moon, and determine the amount of the variation, so far as it depends on this disturbing force.

2. Determine approximately the ratio of the axes of the Moon's oval orbit. What would have been the value of this ratio if the Moon's motion in her orbit had been from east to west.

3. Examine the effect produced on the position of the Moon's nodes by the ablatitious force.

Shew how the plane of a satellite's orbit is affected by the oblateness of the primary.

1846 1. Prove that an orbit similar and equal to the apparent orbit of P round S in motion may be described round S fixed,

by the action of the same central force. NEWTON, *Princip.* Lib. I. Prop. 58.

2. If the equation to the oval orbit of the Moon be

$$r = a\{1 - m^2 \cos(2 - 2m)\,\theta\},$$

and the velocity of the Moon when disturbed by the tangential force, supposing the undisturbed orbit circular, be

$$V\{1 + \frac{3m^2}{4} \cos(2 - 2m)\,\theta\},$$

find the variation.

The law of force in an orbit nearly circular being given, 1847
find an approximate value of the apsidal angle. Newton, Sect. IX.

In Prop. 66 of Newton's 11th Section, shew that the tangen- 1848
tial disturbing force on P always acts towards the nearer syzygy. Is this true of the tangential force on P?

Account for the Moon's variation, as far as it is caused by the tangential disturbing force of the Sun.

In Prop. 66 of Newton's 11th Section, Cor. 11. examine the 1849
effect produced on the motion of the nodes by the ablatitious force.

Shew from general reasoning that in consequence of the oblateness of a planet, the line of intersection of a satellite's orbit with the equator of its primary will continually regrede.

LUNAR THEORY.

1. ASSUMING the equations of motion of the Moon, obtain 1845
the accurate differential equation for calculating s, the tangent of the inclination of the Moon's radius vector to the plane of the ecliptic.

Shew that the integral of $\frac{d^2s}{d\theta^2} + s = 0$ will not be a first approximation to the value of s after several revolutions of the Moon, and find the true first approximation.

2. Explain the physical meaning of the terms

$$2e \sin(cnt - \alpha) + \frac{15}{4} me \sin\{(2 - 2m - c)\, nt - 2\beta + \alpha\}$$

in the expression for the Moon's longitude; and shew why the ratio of the motion of the node to that of the apse is much smaller for the Moon than for one of Jupiter's satellites.

Prove the approximate differential equation to the orbit of the Moon,

$$\frac{d^2u}{d\theta^2} + u - \frac{P}{h^2u^2} + \frac{T}{h^2u^3}\frac{du}{d\theta} + 2\frac{P}{h^2u^2}\int\frac{T}{h^2u^3}\,d\theta = 0.$$

1847 1. Shew that in consequence of the eccentricity of the Earth's orbit, the excess of the Moon's true longitude over the mean is greatest about three months after the Earth has passed its aphelion, and greatest negatively about three months after it has passed its perihelion.

2. Distinguish clearly between the whole force and the disturbing force of the Sun on the Moon, and investigate expressions for the resolved parts of the latter along and perpendicular to the Moon's radius vector.

Calculate approximately the ratio of the difference of the disturbing forces at nearer and further syzygy to the forces themselves.

3. Prove that the centre of gravity of the Earth and Moon describes an orbit round the Sun much more nearly elliptical than that described by the Earth round the Sun.

4. Explain the effect of the term $\frac{1}{8} m^2 \sin\{(2 - 2m)\, nt - 2\beta\}$ in the Moon's longitude.

What important astronomical element may be obtained by comparing the theoretical and observed values of the coefficients of that inequality which depends upon $\sin\{(1 - m)\, nt - \beta\}$, and why is it necessary for this purpose to know the Moon's semi-diameter to a great degree of accuracy?

1848 1. Given in the Lunar Theory

$$\frac{d\theta}{dt} = hu^2\sqrt{1 + 2\int\frac{T}{h^2u^3}d\theta},$$

$u = a\,[1 + e\cos(c\theta - \alpha) + \frac{5}{8}\,me\cos\{(2 - 2m - c)\,\theta - 2\beta + \alpha\}]$, obtain θ in terms of t to the second order.

Find also to the same order the Moon's periodic time in any assigned revolution.

In the Lunar Theory, investigate the differential equation 1849
for the tangent of the Moon's latitude.

PLANETARY THEORY.

1. SHEW that a term in the expansion of the disturbing 1845
function, which goes through all its values in a period nearly equal to that of the disturbed planet, will produce a considerable inequality in the radius vector.

2. Prove the following formula for the variation of the mean distance in a disturbed orbit,

$$\frac{da}{dt} = -\frac{2na^2}{\mu}\frac{dR}{d\epsilon}.$$

Assuming the approximate equations

$$\frac{de}{dt} = \frac{na}{\mu e}\frac{dR}{d\pi}, \qquad \frac{d\pi}{dt} = -\frac{na}{\mu e}\frac{dR}{de},$$

calculate the inequality of the fourth order in the longitude of a disturbed planet arising from the following term in the development of R,

$$Pe^3e'^2\cos\{13\,(nt + \epsilon) - 8\,(n't + \epsilon') - 3\pi - 2\pi'\}.$$

1. Explain the method of integrating by successive ap- 1846
proximations the equation for the perturbation of the radius vector of a planet,

$$\frac{d^2(r\delta r)}{dt^2} + \frac{\mu}{r^3}r\delta r + \Sigma P\cos(pnt - qn't + Q) = 0.$$

Examine the terms which are most increased by the integration, and describe the nature of the forces which produce the corresponding inequalities.

2. When a planet m is disturbed by another m', the variations of the tangent of the inclination and the longitude of the node are given by the equations,

$$\frac{d \tan i}{dt} = m' n a A \tan i' \sin(\Omega - \Omega'),$$

$$\frac{d\Omega}{dt} = m' n a A \left\{ -1 + \frac{\tan i'}{\tan i} \cos(\Omega - \Omega') \right\},$$

where A is the same for both planets; find the form of $\tan \Omega$ in terms of t, and shew what is the condition that the secular motion of the node may be oscillatory.

1847 1. Prove the following equation in the Planetary Theory:

$$\frac{d}{dt}(na^2 \sqrt{1-e^2} + \frac{dR}{d\theta} = 0.$$

Transform the term $\frac{dR}{d\theta}$ so that it may be found immediately from the development of R in terms of t and the elements of the orbits.

2. Investigate expressions for the variations of the inclination and longitude of the node of a disturbed planet. What is the objection to the use of these formulæ when the inclination of the orbit to the plane of reference is very small?

1848 1. In the development of the disturbing function R of a planet which moves in the ecliptic, state and prove the rule which supplies a limit to the dimensions of the co-efficient of a periodic term with a proposed argument.

2. If R be the disturbing function of a planet, prove that

$$\frac{dR}{d\theta} = \frac{dR}{d\omega} - \frac{dR}{d\epsilon},$$

where θ is measured in the plane of the body's undisturbed orbit.

1849 1. Explain how the masses of the planets may be discovered.

2. If Ω be the longitude of the node, I the inclination of the orbit, r the radius vector of a planet, θ its longitude on its orbit, shew that

$$\frac{d\Omega}{dt} = \frac{1}{h \sin I} \frac{dR}{dI},$$

$$\frac{dI}{dt} = \frac{1}{h} \cot(\theta - \Omega) \frac{dR}{dI},$$

where $h = r^2\left(\frac{d\theta}{dt} - 2\sin^2\frac{I}{2}\frac{d\Omega}{dt}\right)$.

3. Given the equations

$$m\sqrt{a}\frac{dp}{dt} = A(q'-q), \quad -m\sqrt{a}\frac{dq}{dt} = A(p'-p),$$

$$m'\sqrt{a'}\frac{dp'}{dt} = A(q-q'), \quad -m'\sqrt{a'}\frac{dq'}{dt} = A(p-p'),$$

where $p = \tan i \sin\Omega$, $q = \tan i \cos\Omega$,

$p' = \tan i' \sin\Omega'$, $q' = \tan i' \cos\Omega'$,

ii' being the inclinations of the orbits of two mutually disturbing planets mm' to the plane of the ecliptic, and $\Omega\Omega'$ the longitudes of their nodes; find the inclination and the longitude of the node of the planet m at any time; and explain the nature of the motion of the node when the minimum inclination is zero.

FIGURE OF THE EARTH.

INVESTIGATE the law of variation of the density in the interior of the Earth supposed at rest, the pressure at any point being assumed to vary as the square of the density. 1845

How is the mean density of the Earth determined?

1. If a variable quantity whose value depends on the direction determined by the angular co-ordinates θ, ϕ be expanded in a series of Laplace's coefficients $Y_0 + Y_1 \ldots + Y_n \ldots$, prove that the particular value of Y_n belonging to a given direction is independent of the position of the line from which θ is measured, and of the plane from which ϕ is measured. 1848

2. Assuming the equation

$$-\frac{\epsilon}{a}\int_0^a \rho' a'^2 da' + \frac{1}{5a^3}\int_0^a \rho'\frac{d}{da'}(\epsilon' a'^5)\,da' + \frac{a^2}{5}\int_a^{a_1}\rho'\frac{d\epsilon'}{da'}\,da' + \frac{a^2\omega^2}{8\pi} = 0,$$

apply it to determine the numerical value of the Earth's ellipticity on the extreme suppositions (1) of a constant density, (2) of a density infinitely greater about the centre than at other points, the centrifugal force at the equator being to the force of gravity as 1 to 289.

3. In a homogeneous fluid mass revolving uniformly about an axis, the oblate spheroid is a form of equilibrium if the force at the pole is to the force at the equator as the equatoreal axis to the polar axis.

State Clairaut's Theorem relative to the figure of the Earth, and mention the hypotheses on which it is obtained. How are the theoretical results in this subject tested by experiment?

1849 **Assuming that the density of the Earth increases from the surface to the centre, shew that the ellipticity of the strata increases from the centre to the surface; the equation for determining the ellipticity of any stratum being**

$$\frac{\epsilon}{a}\int_0^a \rho' a'^2 da' - \frac{1}{5a^3}\int_0^a \rho' \frac{d}{da'}(\epsilon' a'^5)\, da' - \frac{a^2}{5}\int_a^a \rho' \frac{d\epsilon'}{da'}\, da' - \frac{am}{2}\frac{a^2}{a^3}\int_0^a \rho' a'^2 da' = 0.$$

ATTRACTIONS.

1845 **1. Prove that a particle placed in the centre of a uniform plane circular annulus, every particle of which attracts with a force varying inversely as the square of the distance, is in a position of unstable equilibrium.**

How do you account for the stability of Saturn's ring?

2. If the equation to the surface of a nearly spherical homogeneous body of density ρ be

$$r = a\{1 + \alpha(y_0 + y_1 + y_2 + \ldots)\},$$

(where α is a very small quantity and $y_0, y_1, y_2 \ldots$ Laplace's coefficients,) and V be the potential of its attraction on an external particle, the distance of which from the origin is c, prove that

$$V = \frac{4\pi\rho a^3}{3c} + \frac{4\pi\rho a a^3}{c}\left\{y_0 + \frac{y_1}{3}\frac{a}{c} + \frac{y_2}{5}\frac{a^2}{c^2} + \ldots\right\}.$$

Hence shew that, if the attracted particle be on the surface of the attracting body, the expression $V + 2a\frac{dV}{dc}$ has the same value for all bodies differing very slightly in magnitude from the sphere, the radius of which is a.

Shew that a spherical shell of uniform thickness attracts an external particle with the same force as if it were collected at its centre, and that the resultant attraction on an internal particle is zero. 1848

Prove that the attraction of a sphere, whose density is any function of the distance from the centre, on an internal particle is the same as if the mass outside a concentric sphere drawn through the particle were removed, and the mass inside were collected at the centre. 1849

PRECESSION AND NUTATION.

Assuming the motion of the pole of the Earth produced by the Moon's action in a siderial revolution to be $\frac{6\pi^2 B}{T'\omega(n+1)}$ $\sin qP \cos qP$, determine the precessional motion during a long period. 1846

Assuming that the angular velocity of the Earth at any time is the resultant of a uniform angular velocity about the polar axis, and an angular velocity $c \sin\Delta \cos\Delta$ about an axis perpendicular to the plane of the declination circle passing through the Sun, where Δ is the Sun's north polar distance, find the solar precession and nutation. 1847

Hence shew that the true pole appears to describe an ellipse about the mean pole, and find the ratio of its axes. At what times of the year is the obliquity, so far as it depends on the solar nutation, a maximum?

The velocity of the pole perpendicular to the plane passing through the Sun and the Earth's axis being $A \sin 2\Delta$, where Δ is the Sun's north polar distance, find the solar precession. 1849

UNDULATORY THEORY.

1. Two perfectly similar series of waves of common white light of equal intensity emanate from two points A, B, very near each other. Calculate the appearance produced on a sufficiently distant screen, a normal to which meets the line AB at right angles. 1845

Explain how the result of the investigation may be applied to determine the length of an undulation.

2. A plate of Iceland spar is bounded by planes perpendicular to the axis of the crystal: determine the difference of the retardations of the ordinary and extraordinary wave fronts, light being incident nearly in the direction of the axis. Give a general explanation of the origin of the coloured rings produced when a crystalline body is interposed between a polarizing and an analyzing plate.

3. Describe accurately the phenomena of Newton's rings, and how they may be accounted for on the theory of interferences.

If the rings be formed between a prism and a lens, and the angle of incidence at the second surface of the prism exceed the critical angle, what appearance is presented? What conclusion would this seem to warrant respecting the extent of the molecular influence of glass on the luminiferous ether.

1846 1. Explain, without analytical investigation, the formation of dark bands on a screen which intercepts two perfectly similar series of waves of homogeneous light, starting in the same phase from two points very near each other; shewing that the locus of the dark points in space corresponding to any given band is a hyperboloid of revolution.

Shew how this phenomenon may be employed in determining the refractive index of a gas.

2. Apply the principle of least action to prove the law of refraction on the corpuscular theory of light.

3. What evidence is there that light travels more slowly in refracting media than in vacuum? What reason is there for believing that the refrangibility of light depends on the lengths of the waves, but that colour does not depend simply on that circumstance? By what experiments have Fresnel's formulæ for the intensities been verified?

4. Explain the artifice by which the laws of refraction from vacuum into a doubly refracting medium are made to

depend on those of the propagation of plane waves within the medium.

Shew that there are two directions at right-angles to each other, in the front of a plane wave propagated in the interior of a doubly refracting medium, in which if a particle be displaced, the force of restitution resolved parallel to the plane front will be in the direction of displacement.

1. A plane mirror is placed so that its plane nearly passes 1847
through a luminous point; explain the formation of the fringes produced by the interference of the reflected light with that which comes directly from the luminous point.

What important conclusions may be deduced from the result of this experiment, besides those which may be deduced from ordinary cases of interference?

2. Circularly polarized light is transmitted through a plate of Iceland spar cut in a direction perpendicular to the axis, and the emergent light is analyzed; explain the appearance of the rings.

If we assume that Fresnel's expressions give correctly the *change* in the phase of polarized light which suffers total internal reflection, shew that the result of the preceding investigation, compared with experiment, renders it necessary to suppose that change consists in an *acceleration*, the circularly polarized light employed being supposed to be procured by means of a Fresnel's rhomb. Shew also that this conclusion remains the same whether we suppose the vibrations of plane polarized light to be in or perpendicular to the plane of polarization.

3. A plane wave of light is incident obliquely on a moderately small aperture in a screen; find the magnitude of the vibration at any point beyond the screen, considering the propagation of the disturbance in two dimensions only. What objection to the Undulatory Theory is removed by this investigation?

What are the objections to the Newtonian Theory, derived

(1) from the refrangibility of the light of different fixed stars being the same, and unaffected by the motion of the earth, (2) from refraction being accompanied by reflection? How far have satisfactory answers been made to these objections?

1848 1. Light is incident obliquely on a thin plate separating two media, or two portions of the same medium; obtain expressions for the intensities of the reflected and transmitted light.

N.B. The sum of the infinite series

$\sin a + r \sin(a - \beta) + r^2 \sin(a - 2\beta)\ldots$ is $\dfrac{\sin a - r \sin(a + \beta)}{1 - 2r \cos\beta + r^2}$,

r^2 being < 1.

If the co-efficient of vibration be multiplied by b, c for reflection and refraction at the first surface, and by e, f for reflection and refraction at the second, and if the first and third media be the same, shew, that the condition that the sum of the intensities of the reflected and transmitted light shall be equal to that of the incident light leads to the relations

$$b = \mp e,\ cf = \pm (1 - e^2).$$

Assuming that $b = -e$, $cf = 1 - e^2$, explain the formation of the colours seen with such plates. Why are the tints more vivid in the reflected, than in the transmitted light, and why are no traces of colour perceptible unless the plate be extremely thin?

2. Give the theoretical definitions of plane, circularly, and elliptically polarized light: and shew that plane or elliptically polarized light may be resolved into two streams of circularly polarized light, one right-handed and the other left-handed.

It is found that certain liquids possess the property of twisting the plane of polarization of plane polarized light through an angle, which, for a given colour is proportional to the length of the path of the ray within the medium: prove that this property is accounted for, by supposing the medium to transmit right-handed and left-handed circularly polarized light with different velocities, which are the same in all directions.

3. Explain briefly the general nature of an undulation, and the distinction between the propagation of an undulation and the transmission of matter. What are the principal phenomena of light, the explanation of which is independent of the mode of vibration? What arguments may be adduced in favour of the theory of transversal vibrations, and against the existence of normal vibrations? What *a priori* difficulty presents itself with reference to transversal vibrations, and how may it be got over?

1. State and explain Flamsteed's method of determining the 1849
first point of Aries, introducing all the corrections which are required to ensure accuracy in the result.

2. What phenomena of light correspond to the pitch and loudness of a musical note? What other phenomena which may be taken account of by the undulatory theory are peculiar to light and sound respectively?

Explain in general terms the reason why a ray is divided into two polarized rays, on entering into a doubly refracting medium.

3. Explain without integration the formation of spectra by a grating, placed close to the object-glass of a telescope, with which a slit of light parallel to the lines of the grating is viewed: and shew why some of the spectra disappear when the breadths of the transparent and opaque portions of the grating are commensurable.

The distance between the images of the same fixed line in a given pair of the spectra is measured; shew how the length of wave corresponding to that fixed line may thence be determined, giving the requisite formula.

4. A plane wave of light is transmitted obliquely through a plate of Iceland spar cut in a direction perpendicular to the axis, find the difference of retardation of the ordinary and extra-ordinary waves.

Circularly polarized light is transmitted through such a plate, and the emergent light is circularly analyzed; investigate an expression for the intensity of the rings.

DEFINITE INTEGRALS.

1845 If a, a', β, β', be quantities subject to the conditions

$$a^2 + a'^2 = 1, \qquad \beta^2 + \beta'^2 = 1, \qquad a a' + \beta \beta' = 0,$$

then will

$$\int_0^{2\pi} f(a \cos u + a' \sin u, \beta \cos u + \beta' \sin u)\, du = \int_0^{2\pi} f(\cos u, \sin u)\, du.$$

What is meant by the *principal* value of a definite integral which presents itself under an indeterminate form?

1846 Prove that the definite integral

$$\int_{-l}^{l} \frac{(1 - p^2) f(x')\, dx'}{2l \left\{1 - 2p \cos \frac{\pi (x - x')}{l} + p^2\right\}},$$

in which l is a given positive quantity, has $f(x)$ for its limiting value, when p, being always less than 1 approaches indefinitely near to 1, and x lies between $-l$ and $+l$.

Find the value of $\int_0^\infty \frac{\sin x}{x}\, dx$; and thence deduce that of

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx.$$

FINITE DIFFERENCES.

1845 $v_a, v_b \ldots v_c$ being values of v corresponding to values $a, b, \ldots c$ of x, establish the following formula of interpolation,

$$v = \frac{(x - b) \ldots (x - c)}{(a - b) \ldots (a - c)} v_a + \frac{(x - a) \ldots (x - c)}{(b - a) \ldots (b - c)} v_b + \&c.$$

Shew that if $v_a = v_b = \ldots = v_c$, the value of v given by this formula is independent of x.

1847 * Is there any essential distinction between those integrals of an equation of finite differences which have been called direct and indirect?

1848 1. Integrate the equation $u_{x+2} - (a + b) u_{x+1} + ab u_x = \cos x\theta$; and supposing $b > a$ determine the arbitrary constants by the

conditions that $a^{-x}u_x$ shall not become infinite with x, and that $u_0 = 0$.

Find also the complete integral when $b = a$.

2. Prove that $\Sigma^n a^{-x}u_x = a^{n-x}(D-a)^{-n}u_x$; and shew that

$$(D-a)^{-n}\,0 = a^x\{C_1 + C_2x + \ldots + C_nx^{n-1}\}.$$

Shew also that a particular value of $\Sigma^n a^{-x}\cos\theta x$ is

$$\frac{a^{n-x}\,.\,(D^{-1}-a)^n\cos\theta x}{(1-2a\cos\theta+a^2)^n}.$$

1. How may finite integration be applied to the summation 1849
of series?

2. Explain what is by meant $\Delta^m 0^n$; and if $f(e^t)$ can be expanded in a series of ascending powers of t, shew that

$$f(e^t) = f(1+\Delta)\,1 + f(1+\Delta)\,0\,.\,t + \&c. + \frac{1}{\lfloor m}.f(1+\Delta)\,0^m.t^m + \&c.$$

3. Prove the formula

$$\Sigma u_x = \int u_x dx - \tfrac{1}{2}u_x + \frac{B_1}{1\,.\,2}\frac{du_x}{dx} - \frac{B_3}{1.2.3.4}\frac{d^3u_x}{dx^3} + \ldots$$

4. Investigate the equations:

$$u_{x+n} = u_x + n\Delta u_x + \frac{n(n-1)}{\lfloor 2}\Delta^2 u_x + \ldots\ldots$$

$$\Sigma(u_x) = \int u_x dx - \frac{u_x}{2} + \frac{B_1}{\lfloor 2}\frac{du_x}{dx} - \frac{B_3}{\lfloor 4}\frac{d^3u_x}{dx^3} + \ldots\ldots$$

CALCULUS OF VARIATIONS.

1. Shew that, whatever function y is of x, if y and z are so 1845
related that $\phi\left(x, y, \frac{dy}{dz}\ldots\right) = \psi\left(x, z, \frac{dz}{dx}\ldots\right)$, then $A\delta y = B\delta z$, where A is the expression which equated to zero forms the condition of maximum or minimum for $\int\phi dx$, and B is the corresponding expression for $\int\psi dx$.

2. Investigate the condition in order that $\int_b^a \phi\left(x, y, \frac{dy}{dx}\ldots\right)dx$ may be a maximum or a minimum.

3. The shortest distance between two straight lines is perpendicular to both.

1846 1. Explain the principle of the *Variation of Parameters*, as applied to the solution of differential equations.

2. Investigate the variation of $\int f\left(y, \frac{dy}{dx}\right) dx$, the limits of the integral being invariable.

1847 Of all plane isoperimetrical curves, whose extremities are given, shew that the greatest solid of revolution, about a given axis, will be generated by that curve of which the radius of curvature varies inversely as the ordinate.

MAGNETISM.

1846 If V be the potential of the Earth's magnetic force at a place (P) on the Earth's surface, for which the horizontal intensity is ω, θ the angle which an element of a curve ds, drawn through P, makes with the magnetic meridian at P, then

$$dV = \omega \cos\theta\, ds.$$

Hence shew that the value of the integral $\int \omega \cos\theta\, ds$ taken throughout the whole length of a closed line on the Earth's surface = 0.

HEAT.

1845 Investigate the equation for determining the permanent temperature of any part of a prism of small transverse section.

If the two extreme sections of the prism be maintained at constant temperatures, and its length be supposed divided into any number of equal parts, and $v_1, v_2, v_3 \ldots$ be the temperatures at these points of division, prove that $\frac{v_p + v_{p+2}}{v_{p+1}}$ is a constant quantity.

TIDES.

1849 * Explain how the attractions of the Sun and Moon produce the tides; and account for spring tides and neap tides.

VIBRATIONS.

FIND the number of vibrations in one second corresponding to the fundamental note of a straight tube open at one end and closed at the other. 1845

How may the velocity of sound in any gas be measured?

How do you account for the fact of a series of harmonics being sometimes audible when a string is made to sound its fundamental note? 1846

The same musical note is produced in each of two railway trains travelling in the same direction; shew that a person standing on the line between the trains will hear beats, and determine the number of beats per second, when the pitch of the note and the velocities of the trains are given. 1847

SOUND.

IN the eruption of a volcano, it has been remarked that every ejection of stones is accompanied with an explosion like artillery, when heard at a distance; but that when near, the sound resembles rather that of a loud and deep sigh, unaccompanied by any sudden burst; explain this phenomenon, and apply the principle of your explanation to shew that a flash of lightning may produce a lengthened peal of thunder, without the reverberation of the clouds. 1846

Investigate an expression for the velocity of sound in a fluid of any sort. 1848

HYDRODYNAMICS.

WHAT postulate is involved in the ordinary mode of applying D'Alembert's principle to establish the fundamental equation of Hydrodynamics? 1845

A given mass of incompressible fluid acted on by gravity flows through a tube, determine the motion. 1848

Prove the equation of steady motion in Hydrodynamics. 1849

PART III.

EUCLID.

1839 ✓ WHEN can a circle be inscribed in a quadrilateral figure?

1840 If the sides of the triangle be as the numbers 2, 4, 5, shew whether it will be acute or obtuse angled.

1841 Shew that if the points of intersection of the circles, in Euclid's figure, be joined with the vertex of the triangle and with each other, another triangle will be formed similar and equal to the former.

1847 If the four sides of a quadrilateral figure are equal to one another, the diagonals bisect each other at right angles.

1848 1. * If the two diameters be drawn, shew that a parallelogram will be divided into four equal parts. In what case will the diameter bisect the angle of a parallelogram?

2. * Shew that all equal straight lines in a circle will be touched by another circle.

3. * If two straight lines *AEB, CED* in a circle intersect in *E*, the angles subtended by *AC* and *BD* at the centre are together double of the angle *AEC*.

1849 1. * By a method similar to that used in Euc. B. 1. P. 1., describe on a given finite straight line an isosceles triangle, the sides of which shall be each equal to twice the base.

2. * Can you give Legendre's method of demonstrating Euc. B. 1. P. 32., which depends upon the necessary homogeneity of algebraical equations, or any demonstration other than Euclid's?

3. * Shew that in Euclid's figure B. II. P. 7. four other lines, beside the given line, are divided in the required manner.

4. * If a straight line touch a circle, the radius to the point of contact is perpendicular to it;

Give a direct demonstration of this proposition by the method of limits.

5. * How may a circle be described touching one side of a triangle, and the produced parts of the two others?

6. * Apply Euc. B. VI. P. 4., to prove that the rectangle contained by the segments of any chord passing through a given point within a circle is constant.

7. * Prove that equal right lines drawn from a given point to a given plane are equally inclined to the plane.

ARITHMETIC.

1. *A* can do a piece of work in 16 days, and *B* can do it in 20 days; in what time will they do it when they work together. 1838

2. What sum of money must be paid down in order to receive £360. 10*s*. two years hence, allowing $3\frac{1}{2}$ per cent. compound interest?

3. Divide 72.36 by .0036.

4. Extract the square roots of 1.21. and 12.1.

5. Find the side of the cube which contains 15 solid feet and 1080 solid inches.

A man sells a horse for £24. 12*s*. and loses £17. 16*s*. per cent. on what the horse cost him, what was the original cost? 1839

If five men can reap a field whose length is 800 feet and breadth 700 feet in $3\frac{1}{3}$ days of fourteen hours each, in how many days of 12 hours each can seven men reap a field whose length is 1800 feet and breadth 960 feet? 1840

1. State the difference between interest and discount, and find the discount on £397. 6*s*. 8*d*. due 9 months hence, at 4 per cent. per annum. 1841

2. Find the number of feet, inches and parts, in the side of a square whose area is 14 feet 11 inches.

3. Divide 95.88 by 4700 and also by .068.

1842 1. Define discount, and determine the discount on £120. 16*s.* 8*d.*, due ten months hence, at $4\frac{1}{2}$ per cent.

2. Express 2 ft. $10\frac{1}{4}$ in. linear measure, and 5 ft. $79\frac{1}{6}$ in. square measure in the duodecimal scale of notation as feet and duodecimals of a foot; also, the latter quantity being the area of a rectangle one of whose sides is the former, find its other side by dividing in the duodecimal scale.

1843 1. What sum must be invested in the 3 per cent. stock at $94\frac{1}{4}$, to yield an annual income of £500?

2. What is the income corresponding to an income tax of £13. 2*s.* 6*d.*, at the rate of 7 pence in the pound?

1844 1. What sum would be saved annually, if the interest on a public debt of £4.000.000 were reduced from $3\frac{1}{2}$ per cent. to 3 per cent.? If, in consequence, the price of this stock fell from £101. to £$95\frac{3}{8}$., how much would the whole property of the fundholders be diminished?

2. Divide .00071012 by 1.732, and extract the square root of 196.1 to three places of decimals.

3. Reduce the fraction $\frac{1}{72}$ to a duodecimal.

1845 1. A Turkey carpet, measuring 11 ft. 6 in., by 9 ft. 8 in., is laid down on the floor of a room measuring 14 ft. by 12 ft. 6 in., determine the quantity of floor-cloth necessary to complete the covering of the area, and its price at 6*s.* per square yard.

2. Find the value of $\frac{\sqrt{5}}{\sqrt{3}+\sqrt{2}} + \frac{\sqrt{5}}{\sqrt{3}-\sqrt{2}}$ to four places of decimals.

1846 Gunpowder being composed of nitre 15 parts, charcoal 3 parts, and sulphur 2 parts, find how much of each is required in making 16 cwt. of powder.

1847 1. If gold can be beaten out so thin that a grain will form a leaf of 56 square inches, how many of these leaves will make

an inch thick, the weight of a cubic foot of gold being 10 cwt. 95 lbs.?

2. Find by duodecimals the height of a parallelopiped which contains 659 feet 1248 inches, and whose base is 26 feet 6 inches.

* If each inmate of a workhouse cost per week, 2*s.* 2½*d.* for 1848
food, 5½*d.* for clothes and washing, and 3*d.* for lodging, and the number of persons thus maintained be estimated at 100000, what is the whole yearly cost; and how many labourers at 18*d.* a day wages would earn the same sum in the same time?

1. *Shew that the value of a product does not depend upon 1849
the order of its factors, the factors being commensurable. Extend your proof to the case of incommensurable factors.

2. *A person rents a piece of land for £120 a year. He lays out £625 in buying 50 bullocks. At the end of the year he sells them, having expended £12. 10*s.* in labour. How much per head must he gain by them, in order to realize his rent and expenses, and 10 per cent. upon the original outlay?

3. √* A person in London owes another in Petersburgh a debt of 460 rubles, which must be remitted through Paris. He pays the requisite sum to his broker, at a time when the exchange between London and Paris is 23 francs for £1. and between Paris and Petersburgh 2 francs for one ruble. The remittance is delayed until the rates of exchange are 24 francs for £1. and 3 francs for 2 rubles. What does the broker gain or lose by the transaction?

ALGEBRA.

1. FIND the least common multiples of 105, 180, 96, and of 1838
$$6x^2 - x - 1,\ 2x^2 + 3x - 2.$$

2. Find all the positive and integral values of x and y which satisfy the equation $17x + 23y = 200$; and of x, y, z, which satisfy the equation
$$17x + 23y + 3z = 200.$$

3. Divide $a^2 + b^2$ by $a + b\sqrt{-1}$, and $x - y$ by $x^{\frac{1}{m}} - y^{\frac{1}{m}}$: shew that the quotient in the latter case is finite.

1839 1. Find the least common multiple of
$x^2 - 4a^2$, $x^3 + 2ax^2 + 4a^2x + 8a^3$, and $x^3 - 2ax^2 + 4a^2x - 8x^3a^3$.

2. Shew that $\frac{a}{b^2} + \frac{b}{a^2}$ is always greater than $\frac{1}{b} + \frac{1}{a}$.

3. Solve the equations,

$$.15x + 1.575 - .875x = .0625x,$$

$$\text{and } 2\sqrt{x} + \frac{2}{\sqrt{x}} = 5,$$

4. Find in how many different ways the sum of £4. 16*s*. can be paid in guineas, crowns, and shillings, so that the number of coins used shall be exactly 16; and in each case determine the number of guineas, crowns and shillings respectively.

5. An alphabet consists of 10 consonants and 4 vowels, find how many words can be formed consisting of 3 consonants and 2 vowels each.

6. Given log 1.0686 = .02881517, and log 1.0687 = .02885581, find the number of which the logarithm is .02883549.

7. Transfer 34.13 from the scale whose radix is 6 to that whose radix is 7; and find the simplest fraction which is represented by .1515.. in the scale whose radix is 7.

1840 1. Shew that the square root of $a + \sqrt{b}$ may be extracted in the form $\sqrt[4]{\alpha} + \sqrt[4]{\beta}$ when $1 - \frac{a^2}{b}$ is a perfect square.

2. Determine the coefficient of x^4 in $(1 + x + x^2)^{-5}$.

3. Find the coefficient of x^n in the expansion of $\frac{(1+x)^2}{e^x}$, where e is the base of the Napierian system of logarithms.

4. Supposing an urn to contain an equal number of white and black balls, find the probability that in drawing an even number of balls, there will be an equal number of each colour; the probability of drawing any even number whatever of balls being the same.

1841 1. Divide $x^4 - ax^3 + 2a^3x - 4a^4$ by $x^2 - 2a^2$; and find the condition that $ax^2 + bx + c$ may be a complete square.

2. Shew that the sum of a fraction and its reciprocal is always greater than 2. How can we, consistently with this truth, assume that

$$x + \frac{1}{x} = 2\cos\theta.$$

3. Find the least common multiple of $3x^2 - 5x + 2$ and $4x^3 - 4x^2 - x + 1$.

4. Solve the following equations:

$$(1)\ \frac{1}{x-2} - \frac{2}{x+2} = \frac{3}{5}.$$

$$(2)\ \left.\begin{aligned} x - y &= 12 \\ x^2 + y^2 &= 74 \end{aligned}\right\}.$$

5. Find the number of words which can be formed out of 7 letters taken all together, each word being such that 3 given letters are never separated.

6. Find the coefficient of x^r in the expansion of $\left(x - \frac{1}{x}\right)^{2n-1}$.

7. Find the fractions which converge to $\frac{44}{61}\frac{5}{2}$.

If mb and na be positive integral values of x and y respectively, which satisfy the equation $ax + by = c$, shew how to find all the positive integral values, and determine their number.

1. Find the integral and positive values of x and y which satisfy the equation $13x + 19y = 1174$, and shew that the integral solutions of the indeterminate equation $ax + by = c$ are in arithmetical progression. 1842

2. Find the sum of any number of terms of the recurring series $0 + 4 - 16 + 48 - \&c.$

3. Having given $\log_{10} 2 = .301030$, find $\log_{10} .0025$.

1. Find the least common multiple of $x^3 - x$, $x^3 - 1$, $x^3 + 1$. 1843

2. Solve the equations:

$$(1)\ \frac{2x-6}{5} - \frac{x-4}{9} - \frac{3x}{13} = 0. \qquad (2)\ x^4 - 14x^2 + 1 = 0.$$

3. The sum of an arithmetic progression is 72, the first term is 24, and the common difference -4, find the number of terms.

4. Out of a set of cards, numbered from 1 to 10, a card is drawn and replaced; after six such drawings find the probability that the sum of the numbers drawn is 24.

1844 1. Find the greatest common measure of
$4x^4 + 9x^3 + 2x^2 - 2x - 4$ and $3x^3 + 5x^2 - x + 2$.

2. Solve the equations:

$$(x-1)(x-2) = 2,$$

$$x - 1 + 2(x-2) + 3(x-3) + \ldots \text{ to 6 terms} = 14,$$

$$\left.\begin{aligned} yz &= bc, \\ \frac{x}{a} + \frac{y}{b} &= 1, \\ \frac{x}{a} + \frac{z}{c} &= 1. \end{aligned}\right\}$$

3. Find the approximate equated time of paying two debts of £a and £a', due at the times t and t' respectively: the sums having been lent at different rates of interest r and r'.

4. Expand $\frac{1}{1 - x + x^2}$ to five terms by the method of indeterminate coefficients; and shew that, if a_n, a_{n+1}, a_{n+2}, be any three consecutive coefficients, $a_{n+2} + a_n = a_{n+1}$.

5. Apply the test of convergence to the series,

$$1 - \frac{x-1}{x+1} + \frac{(x-1)(x-2)}{(x+1)(x+2)} - \frac{(x-1)(x-2)(x-3)}{(x+1)(x+2)(x+3)} + \&c.$$

6. Find the first four convergents to the value of $\sqrt{10}$ by the method of continued fractions.

7. What is the chance that a person, with two dice, will throw aces at least four times in six trials?

1845 1. Determine the square root of $4x^4 - 4x^3 + 5x^2 - 2x + 1$.

2. Solve the following equations:

$$\frac{x-1}{x+1} = 1 + \frac{1}{x},$$

$$\left.\begin{aligned} x + y &= x^2 \\ 3y - x &= y^2 \end{aligned}\right\},$$

$$\sqrt{a+x} + \sqrt{a-x} = \sqrt{b}.$$

3. If 3, 2, 1 be simultaneous values of x, y, z respectively in the preceding proposition, determine the value of x when $y = 2$ and $z = 4$.

4. Find three simple fractions in arithmetical progression whose denominators shall be 6, 9, 18, and whose sum shall be $2\frac{2}{3}$.

5. There are four numbers, the first three of which are in geometrical progression, and the last three in arithmetical; the sum of the first and last is 14, and that of the second and third 12: find the numbers.

6. The ratio of the number of combinations of $4n$ things taken $2n$ together to that of $2n$ things taken n together is

$$\frac{1 \cdot 3 \cdot 5 \ldots (4n-1)}{(1 \cdot 3 \cdot 5 \ldots 2n-1)^2}.$$

7. In what scale will the number 95 be denoted by 137?

8. A coin is thrown up 20 times in succession; what is the probability that the head will present itself an odd number of times?

1. Prove that 1846

$$(a^2 + ab + b^2)(c^2 + cd + d^2) = (ac + ad + bd)^2 + (ac + ad + bd)(bc - ad) + (bc - ad)^2,$$

and shew that there are two ways in which the above product can be expressed in the form $p^2 + pq + q^2$, and two ways in which it can be expressed in the form $p^2 - pq + q^2$.

2. Solve the equations:

$$\frac{a^3 + x^3}{a + x} + \frac{a^3 - x^3}{a - x} = 4a^2 \ldots\ldots (1)$$

$$(x^2 + 1)(x + 2) = 2 \ldots\ldots (2)$$

$$\left.\begin{aligned} \frac{m}{x} + \frac{n}{y} &= a \\ \frac{n}{y} + \frac{m}{y} &= b \end{aligned}\right\} \ldots\ldots (3)$$

In equations (3), explain the result when the values of x and y assume the form $\frac{0}{0}$.

3. If the expression for the amount be expanded according to powers of r, (the rate of interest), give an interpretation to each term of the expansion.

4. The sum of n terms of an arithmetical series is 88, the first term (a) is 16, and the common difference (b) is 4. Find n; and shew generally that a negative value of n corresponds to a series whose first term is $b - a$, and common difference b.

5. If the terms of the expansion of $(a + b)^n$ be multiplied respectively by the quantities $\frac{n}{r}$, $\frac{n-1}{r^2}$, $\frac{n-2}{r^3}$, ... and n be a whole number, find the sum of the resulting series.

6. Transform 545 from the senary scale to the denary, and 5.75 from the denary to the senary.

1847 1. Why are surd numbers said to be incommensurable? Explain why no integer or terminating decimal can have a recurring decimal for its root. In what case can a recurring decimal have such a root? Shew that

$$\sqrt{\cdot 444 \ldots} = \cdot 666 \ldots$$

2. Solve the equations,

$$(1) \quad \frac{x}{x-1} = \frac{3}{2} + \frac{x-1}{x},$$

$$(2) \quad 3x - 2y = 3y - 4x = 1.$$

3. The sum of n terms of an arithmetic series is $pn + qn^2$ whatever be the value of n; find the m^{th} term.

4. Given $\log_{10} 2 = \cdot 3010300$, find $\log_{10} \sqrt[3]{(\cdot 0125)}$.

5. Find the positive integral values of x and y, which satisfy the equation $99x + 19y = 1900$.

6. Find the two middle terms of the expansion of $(a + x)^{13}$; and expand $\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}$ in ascending powers of x, writing down the n^{th} term both when n is odd and when it is even.

1848 1. * If 25 men do a piece of work in 24 days, working 8 hours a day, in how many days would 30 men do the same piece of work working 10 hours a day.

2. * Find the highest common divisor of $x^3 - x^2 - 2x + 2$ and $x^4 - 3x^3 + 2x^2 + x - 1$.

3. * Solve the equations,

(1) $\dfrac{x+1}{x-1} + \dfrac{x+2}{x-2} = 2\,\dfrac{x+3}{x-3}$.

(2) $\sqrt{1+x+x^2} - \sqrt{1-x+x^2} = mx$.

(3) $x + y - z = 8x + 3y - 6z = 3z - 4x - y = 1$.

Discuss the nature of the roots in (2) as affected by the value of m.

4. * Given that the area of an ellipse varies as either axis when the other is constant, and that the area of a circle of radius unity = 3.14..., find the area of the ellipse whose axes are 3 and 5.

5. * Find the least number of pounds which can be paid in either half-crowns or guineas.

6. * If P be the sum of the series formed by taking the 1st and every p^{th} term of an infinite geometric series whose first term is 1 and whose ratio > 1, Q the sum of the series found by taking the 1st and every q^{th} term, prove that

$$P^q\,(Q-1)^p = Q^p\,(P-1)^q.$$

7. * Write down the general term of the expansion of $(1 - 2x)^{-\frac{7}{2}}$.

8. * Find the coeffieient of x^8 in the expansion of

$$(1 - 2x^2 + 4x^4)^{-2}.$$

9. * Transfor $\sqrt{53}$ into a continued fraction, and find the first four converging fractions.

1. * Solve the equations: 1849

(1) $(a-x)\,\dfrac{x+m}{x+n} = (a+x)\,\dfrac{x-m}{x-n}$.

(2) $x^4 + 4a^3x = a^4$.

(3) $\begin{cases} x^2 + y^2 = a^2. \\ x\cos\alpha + y\sin\alpha = a\sin\beta. \end{cases}$

2. * There are n points in a plane, no three of which are in the same straight line, with the exception of p, which are all in the same straight line; find the number of lines which result from joining them.

THEORY OF EQUATIONS.

1839 1. FORM the equation the roots of which are, 3, 3, 1, − 1.

2. Transform the equation

$$x^3 + 5x^2 + 8x - 1 = 0,$$

into two others which shall each want the third term.

3. Describe the method of approximating to the roots of an equation by means of continued fractions. Apply it to determine the first four converging fractions approximating to the greatest root of the equation

$$x^3 - 3x^2 - 2x + 1 = 0.$$

4. Solve the equation $x^4 + 8x^3 + 6x^2 - 37x - 30 = 0$.

5. Approximate to the greatest root of $x^3 - 7x - 1 = 0$.

6. Find all the values of $(a^3)^{\frac{1}{3}}$. Exhibit the values of the product $(a^3)^{\frac{1}{3}} . (b^3)^{\frac{1}{3}}$ which are possible.

1840 1. Resolve $x^7 + 1$ into its quadratic factors.

2. Find the integer next greater than the greatest root of the equation $x^4 - 5x^3 + 11x - 20 = 0$.

3. If a, b, c be the roots of the equation $x^3 + px^2 + qx + r = 0$, and α, β, γ, the roots of the equation $x^3 - 1 = 0$, find the equation whose roots are all the symmetrical combinations of the form $a\alpha + b\beta + c\gamma$, and express, in terms of its roots, the roots of the original equation.

4. Having given that the equation $f''(x) = 0$ has no root between a and b, $f'(x) = 0$ has one and only one, and that $f(x) = 0$ may either have two or none, shew that $f(x) = 0$ has no real root between a and b unless $a \backsim b$ be greater than the sum of the quantities $\frac{f(b)}{f'(b)}, \frac{f(a)}{f'(a)}$, taken without regard to sign.

1841 1. Take away the second term from the equation

$$x^3 - 6x^2 + 12x + 19 = 0,$$

and find the three roots of the resulting equation.

2. Apply the method of finding the limits of the negative roots to find a negative quantity numerically less than any of the negative roots of the equation $x^3 + 7x + 1 = 0$ and a near limit to them.

1. Solve the equation $x^4 + 3x^3 + 2x^2 + 9x - 3 = 0$, of which 1842
two roots are equal but of opposite signs.

2. Apply Sturm's Theorem to the equation $x^3 - 3x - 4 = 0$.

Decompose $\sin x - 1 = 0$ into its factors. 1843

1. Determine the relation between the coefficients of an 1844
algebraic equation, in order that it may be possible to take away the second and third terms by the same transformation.

Shew that the equation

$$x^3 + 4x^2 + 6x + 3 = 0$$

does not satisfy the condition, but that it will if multiplied by x; and solve the equation.

2. If $f(x) = 0$ be an algebraic equation of the m^{th} degree, then the sum of the n^{th} powers of the roots of the equation will be expressed by $-nP_n$, where P_n is the coefficient of x^{-n} in the expansion of Nap. log. $\frac{f(x)}{x^m}$.

Find the sum of the inverse 4^{th} powers of the roots of the equation $x^2 + ax + b = 0$.

1. Solve the equation $x^4 - 32x + 20 = 0$. 1845

2. By the property $\Sigma(a^2) = p_1^2 - 2p_2$, determine the nature of the roots of the equation $x^3 + 4x^2 + 9x + 1 = 0$.

1. Form the equation, the four roots of which $(\alpha, \beta, \gamma, \delta)$ are 1846

$$\frac{1 + \sqrt{3} \pm \sqrt{2\sqrt{3}}}{2} \text{ and } \frac{1 - \sqrt{3} \pm \sqrt{-2\sqrt{3}}}{2},$$

and thence prove that $\frac{\alpha^2 + \beta^2}{\alpha\beta} + \frac{\alpha^2 + \gamma^2}{\alpha\gamma} + \ldots = 0$.

2. Apply Euler's method of solving biquadratic equations to the following

$$x^4 + 8x^2 - 16x + 20 = 0.$$

3. Solve the equation $\frac{x^n - 1}{x - 1} = 0$, where n is a prime number; and explain what is meant by a *cyclical* arrangement of its roots.

Exemplify this arrangement in the case of $\frac{x^5 - 1}{x - 1} = 0$.

1847 1. If $\sqrt{a} + \sqrt{b}$ be one root, where $\sqrt{a}$ and $\sqrt{b}$ do not involve the same surd part, the three other values obtained by changing the signs of $\sqrt{a}$ and $\sqrt{b}$ will also be roots of the equation.

2. Find all the roots of the following equation, which contains equal roots,

$$8x^4 + 4x^3 - 18x^2 + 11x - 2 = 0.$$

3. Solve the equation

$$x^5 - 3x^4 - 9x^3 + 21x^2 - 10x + 24 = 0.$$

1848 1. Find all the roots of the equation

$$x^5 - 2x^4 - 19x^3 - 19x^2 - 2x + 1 = 0.$$

2. If p be the greatest coefficient of an equation without regard to sign and q the last term no root lies between

$$+\frac{q}{p+q} \text{ and } -\frac{q}{p+q}.$$

TRIGONOMETRY.

1839 INVESTIGATE a method of approximating rapidly to the true value of π; and shew what calculations will suffice to make the result accurate as far as the sixth decimal place.

1840 Expand $(\sin\theta)^7$ in terms of the sines of multiples of θ.

1841 1. Two sides of a triangle being 1 foot and 1000 feet respectively, and the angle opposite to the latter 30°, give the several logarithmic formulæ for obtaining the angle opposite to the other, and explain which is the best.

2. The diameter of the base of a right cone being 1 foot and the vertical angle 12′.45″, find approximately the number of feet in the length of the axis.

1842 Apply the formula

$$(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{1}{m}} = \cos\frac{2r\pi + \theta}{m} + \sqrt{-1}\sin\frac{2r\pi + \theta}{m}$$

to find the three values of $(+1)^{\frac{1}{3}}$.

1843 1. What is the value of tan 195°?

2. When $C = 30°$, $b = 16$, $c = 8$, is the triangle ambiguous or not?

3. Given cosine $34°.29' = .4338335$, cosine $34°.30' = .4335938$: find the cosine of $34°.29'.48''$.

4. Describe the vernier. If the limb of the instrument is divided to $10'$, and the vernier reads to $10''$, how many divisions are there in the vernier?

1. Investigate a general formula for all angles which satisfy the equation 1844

$$\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = 4.$$

2. Expand $(\cos\theta)^8$ in sines and cosines of multiples of θ.

3. Given $\log_{10} 12440 = 4.0948204$,
and $\log_{10} 12441 = 4.0948553$,
find $\log_{10} 12440.345$.

1. Simplify the expression $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ in the case 1845 of an equilateral triangle.

2. Prove that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$.

1. Find $\tan 165°$ from the known value of $\tan 330°$. 1845

2. Find the values of the angle A in the following example: $\sin B = .25$, $a = 5$, $b = 2.5$.

1. Find all the solutions of the following equations: 1846

(1) $\sin 7\theta - \sin\theta = \sin 3\theta$. (2) $\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{2^2}\sin 3\theta \ldots = \frac{4}{5}$.

2. In the formula for plane triangles,

$$B' = \frac{b}{a} \cdot \frac{\sin C}{\sin 1''} + \frac{b^2}{a^2} + \frac{\sin 2C}{\sin 2''}.$$

If $\frac{b}{a}$ be so small that the first two terms of the series will give B with sufficient accuracy, adapt the formula to logarithmic computation.

1. Write down the formulæ for $\sin A$ and $\cos A$ in terms 1847 of $\sin 2A$ which apply when A lies between $135°$ and $225°$.

2. Find all the solutions of the equation

$$\cos 3x + \sin 3x = \frac{1}{\sqrt{2}}.$$

3. Prove that

$$x \sin\theta + \frac{x^2 \sin 2\theta}{1.2} + \frac{x^3 \sin 3\theta}{1.2.3} + \ldots = \epsilon^{x\cos\theta} \sin(x \sin\theta).$$

1848 1. * Given $\tan\frac{A}{2} = 2 - \sqrt{3}$, find $\sin A$ and thence A.

2. * Two sides of a triangle are equal to 3 and 12 respectively, and the contained angle is equal to 30^0; find the hypothenuse of an equal right-angled isosceles triangle.

3. * In determining the height of a mountain by observations at two stations in the same horizontal plane, the distance between the stations being known, if the stations are in the same vertical plane passing through the summit, and the summit (S) is observed from the further station, but a lower point (S') is observed from the nearer, shew that the height determined by the process lies between the heights of S and S'.

4. Given

$$\log 7 = \cdot 8450980,$$
$$\log 58751 = 4\cdot 7690153,$$
$$\log 58752 = 4\cdot 7690227,$$

find $\sqrt[5]{\cdot 07}$ to 7 significant figures.

5. Find the limit of $\left(\cos\frac{\theta}{n}\right)^{n^2}$ for $n = \infty$.

1849 1. * Construct the angle whose tangent is $3 - \sqrt{2}$.

2. * Find $\sin 9^0$ and $\sin 81^0$.

3. * The ratio of two sides of a triangle is 9 : 7, and the included angle is $47^0\ 25'$, find the other angles.

* Given

$$\log 2 = \cdot 3010300,$$
$$\text{L tan } 66^0\ 17'\ 30'' = 10\cdot 3573942,$$
$$\text{L tan } 15^0\ 53' = 9\cdot 4541479\text{; diff. } 1' = 4797.$$

4. * If the sides of a quadrilateral figure inscribed in a circle taken in order are 3, 3, 4, 4, find the area, and the radii of the inscribed and circumscribed circles.

5. * The angular elevation of a tower at a place A due south of it is 30^0, and at a place B, due west of A, and at the distance

a from it, the elevation is 18°: shew that the height of the tower is $\frac{a}{\sqrt{2\sqrt{5}+2}}$.

6. The sum of any number of angles

$$\sin^{-1}\frac{2ab}{a^2+b^2},\ \sin^{-1}\frac{2a'b'}{a'^2+b'^2},\ \&c.$$

may be expressed in the form $\sin^{-1}\frac{2mn}{m^2+n^2}$, where m and n are rational functions of a, b, a', b', &c.

SPHERICAL TRIGONOMETRY.

1840 If the angle C of the triangle ABC be a right angle, shew that the spherical excess

$$= 2\tan^{-1}\left(\tan\frac{a}{2}\tan\frac{b}{2}\right).$$

1842 If a, b be the sides, c the hypothenuse of a right-angled spherical triangle, $\cos c = \cos a \cos b$. Thence deduce the corresponding property of right-angled plane triangles.

1844 1. If two angles of a spherical triangle be respectively equal to the sides opposite to them, prove that the remaining side is the supplement of the remaining angle.

2. In a spherical triangle, if C, c remain constant whilst a and b receive the small increments δa and δb respectively, shew that

$$\frac{\delta a}{\sqrt{1-n^2\sin^2 a}}+\frac{\delta b}{\sqrt{1-n^2\sin^2 b}}=0, \text{ where } n=\frac{\sin C}{\sin c}.$$

1845 Shew how to apply Legendre's theorem, if two sides and the included angle of the triangle are given.

1846 If DE be an arc of a great circle bisecting the sides AB, AC of a spherical triangle in D and E, P the pole of DE, and PB, PD, PE, PC be joined by arcs of great circles, shew that $\angle BPC = 2\angle DPE$.

1847 A cube is turned round one of its diagonals through 180°:

find the inclination of the plane of one of the faces to the plane with which that face coincided in its original position.

1848 1. A lune is formed by two great circles which intersect at right angles; prove that from any point in one of the circles two arcs of great circles can be drawn to the other cutting it at equal angles.

2. * If PM be the ordinate at P, and T the intersection of the tangent at P with the axis, $TP \cdot TY = TM \cdot TS$.

3. * For what position of P is the angle SPH greatest.

4. * Prove that such parallelograms have the least area of all which circumscribe the ellipse.

5. * When is the square of the sum of the conjugate diameters of an ellipse the least?

6. * The chords of curvature through the centre and focus are in the ratio of AC to CP.

1849 1. * If SVs, TVt be two tangents cutting one asymptote in the points S, T, and the other in s, t, prove that

$$VS : Vs :: Vt : VT.$$

2. * The foci of all parabolic sections which can be cut from a given right cone lie upon the surface of another cone.

GEOMETRY OF TWO DIMENSIONS.

1838 1. DRAW a tangent at the extremity of the latus rectum of the ellipse of which the equation is

$$\frac{x^2}{9a^2} + \frac{y^2}{4a^2} = 1.$$

2. Having given the equation to an ellipse $y^2 = \frac{b^2}{a^2}(2ax - x^2)$, find its equation when referred to axes which intersect in the center of the ellipse, and make equal angles with the axis major.

3. Determine the position of the vertex and the direction of the axis of the parabola the equation of which is

$$y^2 + 4xy + 4x^2 + 3ax + a^2 = 0.$$

1. Find the equation to the straight line which passes 1839
through the point (1, 3), and makes an angle of 45° with the straight line whose equation is $2y - x + 1 = 0$.

2. Find the nature, magnitude, and position of the curve whose equation is $y^2 - 2xy + 2x^2 - 2y - 4x + 9 = 0$.

3. A section of a right cone is made by a plane parallel to the slant side; when the plane passing through its directrix, and the vertex of the cone is perpendicular to the cutting plane, determine the vertical angle of the cone.

1. Draw a normal at the extremity of the latus rectum of a 1840
parabola whose equation is $y^2 = 4a(x - a)$; and find its distance from the origin of co-ordinates.

2. Shew that the locus of the extremities of all the polar subtangents to any conic section, the focus being the pole, is a straight line perpendicular to the axis major.

1. Find the equation to a straight line passing through a 1841
given point in the axis of x, and making an angle of 45° with the straight line whose equation is $\frac{x}{a} - \frac{y}{b} = 1$.

2. In the equation $ay^2 + bxy + dy + ex = 0$, find the relation among the coefficients that it may represent two straight lines.

3. Find the equation to the circle of curvature at the extremity of the latus rectum of a parabola.

Find the equations to the asymptotes of the curve $(x - 2a)$ 1842
$y^2 = x^3 - a^3$, and shew that the curve cuts two of the asymptotes at points whose co-ordinates are

$$-\frac{a}{3}, \quad \pm\frac{2a}{3}.$$

1. Find the equations to lines passing through a given 1843
point, and inclined at an angle of 45° to the line $y = x + a$.

2. Find the points of contact of tangents, the perpendiculars on which from the foot of the directrix, shall be equal to one fourth of the latus rectum.

3. Determine the equations to the asymptotes of the curve

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0,$$

shewing what is the necessary condition that it may have asymptotes.

4. If x_0, y_0 be the portions of the axes cut off by a tangent at an apse, r the distance of the point of contact from the origin, shew that $\frac{1}{x_0^2} + \frac{1}{y_0^2} = \frac{1}{r^2}$.

1844 1. Find the area of the figure included by the following lines,

$$x - y = 0, \quad x + y = 0, \quad x - y = a, \quad x + y = b.$$

2. If a chord of the parabola $(y^2 = 4mx)$ be a tangent of the parabola $\{y^2 = 8m\ (x - h)\}$, shew that the line $(x = h)$ bisects that chord.

3. Determine the centre of the curve represented by the equation $x^2 - 4xy + 4y^2 - 2ax + 4ay = 0$.

1845 1. If the axes are rectangular, what angle do the two straight lines $x + y\sqrt{3} = 0$ and $x - y\sqrt{3} = 2$ make with each other?

2. What is the eccentricity of the ellipse whose equation is $2x^2 + 3y^2 = c^2$?

3. If a cone be right-angled, what is the inclination of its axis to the planes of those sections the eccentricity of which is $\frac{1}{\sqrt{2}}$?

4. Prove that every straight line which touches a curve of the third order must also cut the curve. In what case will it cut it at the point of contact?

1846 1. The equations of two straight lines, referred to oblique axes inclined at an angle ω, being $y - mx = 0$ and $my + x = 0$, shew that the angle between them is $\tan^{-1}\left(\frac{m^2 + 1}{m^2 - 1}\tan\omega\right)$.

2. Two parabolas have a common focus and axis, and a tangent to the one parabola intersects a tangent to the other at right angles; find the locus of the point of intersection.

3. From the property $PF \cdot CD = AC \cdot BC$, in an ellipse, combined with the polar equation from the centre, find the length of the perpendicular from the centre upon a tangent, in terms of the angle which it makes with the axis major.

Is it true that all parallelograms circumcribed about an ellipse are equal?

4. If normals at P and D intersect in K, shew that KC is perpendicular to PD.

5. If through any point P of a conic section two chords PQ, PR be drawn, parallel to two fixed straight lines, and making angles α and β respectively with the tangent at P, shew that the ratio PQ cosec α : PR cosec β is constant.

1. Shew how an ellipse whose semi-axes are 4 and 5 1847
inches may be described mechanically.

2. From the equation to the tangent in terms of the angle it makes with the major axis, prove that the locus of the intersection of two tangents to an ellipse, at the extremities of conjugate diameters, is an ellipse similar to the given one.

1. Shew that the locus of a point equidistant from two 1848
given straight lines which meet is two straight lines at right angles to each other, and passing through the point of intersection of the two former.

2. Shew that if a straight line pass through two points whose polar co-ordinates are $r_1 a_1$, $r_2 a_2$, its equation may be put under the form

$$\frac{\sin(a_2 - a_1)}{r} + \frac{\sin(\theta - a_2)}{r_1} + \frac{\sin(a_1 - \theta)}{r_2} = 0.$$

3. In the curve whose equation is $Ax^2 + By^2 = 1$, prove that the equation to the tangent at the point (x, y) is $Ax\xi + By\eta = 1$; and shew that this equation will equally represent the chord of contact for two tangents drawn from a given point, and the locus of the point of intersection of pairs of tangents drawn at the extremities of a chord which always passes through a given point.

In the two latter cases, how do you interpret the result when the given point is the origin, the signs of A and B being supposed different in the former of the two?

4. What is represented by the equation $(x - a)\, y = 0$?

5. $u_1 = 0, u_2 = 0, u_3 = 0, u_4 = 0$ are the equations to the sides of a quadrilateral, $u_1 u_3 + u_2 u_4 = 0$ is the equation to the circumscribing conic sections; shew that the conditions, in order that this may be a circle, satisfy the condition that the sum of the opposite angles of the quadrilateral shall be equal to two right angles.

1849 1. Apply the expression for the distance between two points in the same plane, in terms of their co-ordinates, the axes being oblique to shew that the equation of a circle, radius r, referred to tangents inclined at an axle a as axes, is

$$x^2 + y^2 + 2xy \cos a - 2(x + y)\, r \cot \frac{a}{2} + r^2 \cot^2 \frac{a}{2} = 0.$$

2. Apply the polar equation to the tangent at any point of a conic section to prove, that the locus of the point of intersection of the tangent, with the perpendicular upon it from the focus, is either a straight line or a circle.

DIFFERENTIAL CALCULUS.

1838 1. DIFFERENTIATE the following functions:

$$\log_e \sqrt{\frac{1+x}{1-x}} \text{ and } x^x.$$

2. Determine the value of the fraction $\dfrac{2x - a - \sqrt{2ax - a^2}}{x - a + \sqrt{x^2 - a^2}}$ when $x = a$.

3. Construct the curve of which the equation is

$$y = \sin x - \tfrac{1}{2} \sin 2x + \tfrac{1}{3} \sin 3x - \&c.$$

4. If the equation to a curve be $r = \dfrac{a(1 - e^2)}{1 + e \cos \theta}$; find the tangent of the angle between the radius vector and the tangent when $r = a(1 - e^2)$.

1. Find the value of $\frac{x^3 (\cot 2x)^3 + \sin x}{x}$ when $x = 0$, and 1839
the maximum value of $\left(\frac{a}{x}\right)^x$.

2. Shew that the tangent of the angle at which a curve referred to polar co-ordinates is inclined to the radius vector at any point, is constant in the curve $r = ae^{n\theta}$.

1. Find the value of $x^{x^{1+x}}$ when $x = 0$, and investigate the 1840
maximum and minimum values of $(1 + x^{\frac{1}{3}})(7 - x)^{\frac{2}{3}}$.

2. Inscribe the greatest cone in a given sphere.

3. Determine the rectilinear and hyperbolic asymptotes of the curve whose equation is $y^4 - 4ay^2x + 4byx^2 - x^4 = 0$, and the directions of the branches at its multiple point.

4. Eliminate z from the equations $\frac{d^2x}{dz^2} = \phi(x, y)$; $\frac{d^2y}{dz^2} = \psi(x, y)$.

1. Differentiate $\log_e \frac{x - \sqrt{a^2 - x^2}}{x + \sqrt{a^2 - x^2}}$. 1841

2. Determine the maximum and minimum values of $\frac{(a - x)^3}{(a - 2x)^2}$.

3. Apply Maclaurin's theorem to expand $\log_e \frac{a - x}{a - 2x}$ in series—(1) of ascending, and (2) of descending powers of x, each to three terms.

4. If $(a^2 - x^2)\frac{d^2z}{dx^2} + \frac{a^2 - 3x^2}{x}\frac{dz}{dx} - z = 0$, and $x^2 + y^2 = a^2$, find the differential equation between z and y, y being the independent variable.

1. Differentiate 1842

$(x + \sqrt{1 + x^2})^{\frac{m}{n}}$, $\log_e(\tan x)$, and $\sin^{-1}\left(\frac{x - 1}{\sqrt{2}}\right)$.

2. Find the value of $\frac{m \sin\theta - \sin m\theta}{\theta(\cos\theta - \cos m\theta)}$ when $\theta = 0$, and that of $x^{\frac{1}{1-x}}$ when $x = 1$.

1843 1. Find $\frac{dy}{dx}$ from $y\epsilon^{ny} = ax^m$, and from

$y = \cos^{-1}\sqrt{\left(\frac{a^2 - x^2}{b^2 - x^2}\right)}$; and if $y = \cos mx$, find $\frac{d^n y}{dx^n}$.

2. If (x, y, z), (x', y', z') be two sets of co-ordinate rectangular co-ordinates, (u) any function of them: shew that

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = \frac{d^2u}{dx'^2} + \frac{d^2u}{dy'^2} + \frac{d^2u}{dz'^2}.$$

1844 1. Differentiate $\frac{1+x}{1+x^2}$, and $e^x(1 - x^3)$. If $\sin(xy) = mx$, find the value of $\frac{dy}{dx}$.

2. Verify the equation, $\frac{d^2u}{dxdy} = \frac{d^2u}{dydx}$, and verify this in the case of $u = y\log(1 + xy)$.

3. Investigate the expression for the polar subtangent, and apply it to the curve whose equation is $r = \frac{ab}{ae^{\theta} + be^{-\theta}}$.

4. Trace the curve whose equation is $y = x^2\frac{x - a}{x^2 + b^2}$, and find where it meets its asymptote.

1845 1. Differentiate $\frac{1}{x + \sqrt{1 - x^2}}$, $\log_a a$, $\tan^{-1}\sqrt{1 - x}$.

2. Transform the expression $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2$ into one in which y is the independent variable.

3. Transform $x\frac{du}{dx} + y\frac{du}{dy} + z\frac{du}{dz} + \ldots$. into an expression in which $x, \frac{y^2}{x}, \frac{z^2}{x}, \ldots$. shall be the independent variables.

4. The circle of curvature at any point of a parabola, except the vertex, cuts the axis at two points on opposite sides of the vertex.

5. How is the existence of a conjugate point indicated analytically? Determine whether there is such a point in the curve whose equation is $y^2 - 2xy + 2x^2 - x^3 = 0$.

Trace the locus of the equation $e^{\left(\frac{y}{a}\right)^2} = \sin\left(\frac{x}{a}\right)$.

1. Differentiate the following functions: 1846

$$(a^2 + x^2)\tan^{-1}\frac{x}{a}, \quad e^{x^x}, \quad \frac{x - \sin^{-1}x}{(\sin x)^3},$$

and find the value of the last when x is indefinitely small.

2. Determine the position of the two lines drawn at right angles to each other from the centre to the circumference of an ellipse, the product of which is greatest.

3. Prove that corresponding to a point of maximum or minimum curvature in any curve there is a cusp in its evolute. Is the converse true?

4. Find the asymptotes and points of inflexion in the curve whose equation is $r = a\frac{\theta}{\sin\theta}$; and trace the curve.

5. Trace the curve defined by the equation

$$x^2(x + y) = a^2(x - y).$$

6. A straight line which cuts two fixed lines so as always to include with them the same area will constantly touch a hyperbola of which the two fixed lines are the asymptotes.

1. Differentiate the following functions: 1847

$$\log_e\frac{\sqrt{a^2 + x^2} + x}{a}, \quad \frac{\tan^3\theta}{3} - \tan\theta + \theta, \quad \text{and} \quad \left(\frac{x}{n}\right)^{nx}.$$

2. Find the limiting values of $\frac{\sin^3 x}{x - \frac{1}{2}\sin 2x}$ when $x = 0$, and $\frac{\log x}{x}$ when $x = \infty$.

3. Tangents to a curve of the n^{th} order pass through a given point; find the equation to the curve of the $(n-1)^{\text{th}}$ order which passes through the points of contact.

4. Circles are described on successive double ordinates of a parabola as diameters; shew that their envelope is an equal parabola. What part of this system of circles does not admit of an envelope?

5. Trace the curve $y^2 = x^2 \cdot \frac{a + x}{a - x}$.

6. If $y = a \sin \frac{x}{a}$ be the equation to a curve, shew that the equation to the asymptotes of its evolute will be $x + y \cos n\pi = n\pi a$, n being any integer.

7. $a, b, c, \ldots$ are not proportional to $\alpha, \beta, \gamma, \ldots$ from the property that the ratio of $a + b + c \ldots$ to $\alpha + \beta + \gamma + \ldots$ lies between the greatest and least of the ratios $a : \alpha$, $b : \beta$, $c : \gamma$.

Deduce the theorem that $\frac{f(a+h) - f(a)}{F(a+h) - F(a)}$ is equal to the value of $\frac{f'(x)}{F'(x)}$ for some value of x lying between a and $a + h$, where $f(x)$, $F(x)$ and their derivatives are supposed to be finite and continuous for all values of x between a and $a + h$.

8. How are the dimensions of differential coefficients to be taken?

INTEGRAL CALCULUS.

1838 1. INTEGRATE the following functions,

$$\frac{x}{(x^2+1)(x^2+x+1)^2}, \quad (a^2 - x^2)^{\frac{5}{2}}, \text{ and } (\tan x)^7 \text{ from } x = 0 \text{ to } x = \frac{\pi}{4}.$$

$$\frac{1}{\sqrt{a^2 - 2ax}}; \quad \frac{1}{x^3\sqrt{a^2 - x^2}} \text{ and } \frac{1}{(\sin x)^4 . (\cos x)^2}.$$

2. Find $\int_n x^n e^{-x}$ between the limits $x = 0$ and $x = \infty$.

1839 Find the values of

$$\int \frac{1}{a^4 - x^4} \left(\int_{n=\frac{\pi}{2}} - \int_{n=0}\right) (1 + e \cos x)^{-1}, \quad \frac{x^3}{(a + bx + cx^2)^{\frac{3}{2}}}.$$

1840 Find the values of

$$\int_{-2rc}^{+2rc} \frac{2c^2 + x}{(r^2 + c^2 + x)^{\frac{3}{2}}}\, dx, \int \frac{\sqrt{1 - x^2}}{x^4} \sin^{-1} x\, dx,$$

$$\int \frac{dx}{x^3 - a^3}; \quad \int_0^1 \left(\log \frac{1}{x}\right)^n dx.$$

1841 1. Find $\int_0^a (a^2 - x^2)^{\frac{3}{2}}\, dx$.

2. Integrate the expressions:

$$\frac{x^3 + 3x + 1}{(x+1)^2(x^2 + 2x - 1)}, \quad \frac{\sin^2 x}{a + b\cos^2 x}.$$

3. Find the length of the evolute of a portion of the hyberbola included between the vertex and the extremity of the latus rectum.

1. Integrate $\frac{1}{\sqrt{1 + 2x - x^2}}$, $\frac{3x - 1}{x^3 - x^2 x e^{mx^2} \cdot 2x}$, 1842

$e^{ax}\cos nx$, and then deduce $x^r e^{\quad}\cos nx$.

2. Find the area contained by all the loops of the curve whose equation is $r = a \cdot \sin n\theta$.

3. If $x = u\sin a + v\cos a$ and $y = u\cos a - v\sin a$, prove that

$$\iint f(x, y)\frac{dx\,dy}{\sqrt{1 - x^2 - y^2}} = \iint f_1(u, v)\frac{du\,dv}{\sqrt{1 - u^2 - v^2}}.$$

1. Integrate $\frac{dx}{(a^2 + x^2)^{\frac{3}{2}}}$, $\frac{x\,dx}{(x^2 + a^2)\sqrt{(x^2 + 5a^2)}}$, $\frac{dx}{5 + 3\cos x}$. 1843

2. Trace and find the whole area of the curve

$$r = a(2\cos\theta + 1).$$

1. Transform $\int V dx_1\,dx_2\,dx_3\,dx_4$ to polar co-ordinates, when 1844

$x_1 = r\sin\theta\cos\phi$, $x_3 = r\cos\theta\cos\psi$,

$x_2 = r\sin\theta\sin\phi$, $x_4 = r\cos\theta\sin\psi$,

2. Integrate the following differentials:

$$\frac{dx}{\cos x + \sin x}: \quad \frac{x^2 - 3x + 1}{(x-1)(x-2)}\,dx:$$

and find the value of $\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^3\theta\,d\theta$.

1. Integrate $\frac{dx}{x\sqrt{x + a}}$, $\frac{x^3 dx}{(x-a)^2}$, $\sqrt{a + b\sec^2 x}\,dx$. 1845

2. Determine the volume of the solid generated by the revolution of the curve $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$ about the axis of x, a being greater than b. And shew what the result becomes when a is equal to b.

3. Integrate the following differentials $\frac{dx}{\cos x + \sin x}$, $\frac{x^3 - 3x + 1}{x - 1 \cdot x - 2}\,dx$; and find the value of $\int_0^{\frac{\pi}{2}} \sin^2\theta \cdot \cos^3\theta\,d\theta$.

1846 Determine whether the area included between the curve whose equation is $x^2(x+y) = a^2(x-y)$ and its asymptote is finite or infinite.

1847 Shew that $4a^2$ = the whole area of the curve whose equation is $y^2 = x^2 \cdot \frac{a+x}{a-x}$, the area being supposed to be bounded on one side by the asymptote.

1848 1. Write do[redacted] different[redacted] coefficients of $\sin^{-1}x$ and $\text{versin}^{-1}x$; and by integrating [redacted]lts obtain the expansions of the functions in series according to ascending powers of x. Within what limits of x are these series convergent?

2. The curve whose equation is $(y^2 - b^2)^2 = a^3x$ revolves round the axis of y, find the volume of the closed portion of the solid generated.

3. Find $\int_0^\infty \epsilon^{-cx} \sin ax\, dx$, c being positive; and thence prove, by differentiating with respect to c, that

$$\int_0^\infty \epsilon^{-cx} \sin ax \cdot x^n\, dx = 1 \cdot 2 \ldots n\, (c^2 + a^2)^{-\frac{n+1}{2}} \sin\left\{(n+1)\tan^{-1}\frac{a}{c}\right\},$$

$\tan^{-1}\frac{a}{c}$ denoting the smallest angle, without regard to sign, whose tangent is $\frac{a}{c}$.

4. If $u = a^{x^y} + \sqrt{\sec xy}$, find $\frac{du}{dx}$, (1) when x and y are independent, (2) when they are connected by the condition

$$x + y = a.$$

If the equation connecting x and y be $a^{x^y} + \sqrt{\sec xy} = 0$, find $\frac{dy}{dx}$.

5. Find the limiting value of $\frac{\epsilon^x + \epsilon^{-x} - 2\cos x}{x^2}$ when $x = 0$.

6. Trace the curve $(y^2 - b^2)^2 = a^3x$;

7. If two curves, capable of revolving in one plane about two fixed points respectively, are such that they will roll in continuous contact, shew how to determine the equation to

either curve when that to the other is given; and apply the result to the case in which the given curve is a logarithmic spiral revolving about its pole.

1. Deduce $\int \frac{dx}{\sqrt{a^2+x^2}}$, from $\int \frac{dx}{\sqrt{a^2-x^2}}$. 1849 CC.

2. Shew how to change the independent variable in a differential coefficient from x to a given function of x. CC

Given $u = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2$, $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$;

prove that

$$u = \left(\frac{ds}{dt}\right)^{-4}\left\{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 - \left(\frac{d^2s}{dt^2}\right)^2\right\}$$

3. Prove that $\phi(fx)$ is necessarily either a maximum or a minimum when $f(x)$ is a maximum.

Find the greatest cone which can be generated by a right-angled triangle of given hypothenuse.

4. Find the envelope of a system of concentric ellipses in which the directions, and the sum of the squares of the axes are given; and interpret the result.

5. Trace the curve $x^2y^2 + (x^2 - a^2)(x^2 - b^2) = 0$; and shew that the breadth of each closed portion is twice as great in the direction of y as in that of x. Prove also that when b approaches to a as its limit, each of these portions is ultimately similar to an ellipse.

6. Find the difference of the areas cut off by lines joining their consecutive points of inflexion.

GEOMETRY OF THREE DIMENSIONS.

1. If a surface of the second order be cut by a plane passing through a fixed point, the locus of the centres of the plane sections is a surface of the second order, and is similar to the proposed surface. 1839

2. Find the volume of the portion of an ellipsoid contained between any two planes inclined at a given angle and passing through the axis of z.

3. The surface of a right cone is intersected by that of a right cylinder, whose base is an equiangular spiral, having its pole in the vertex and its plane perpendicular to the axis of the cone; shew that the curve thus formed cuts the generating line of the cone at a constant angle, and has for one of its involutes another equiangular spiral in the same plane with the former.

4. Shew how to determine when a curve defined by two equations between [redacted]-ordinates is a plane curve.

Find the relation between [redacted] when the equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad \text{and} \quad \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{xy}{\gamma^2}=1,$$

represent a plane curve.

1841 1. Find the equation to a plane which passes through the point whose co-ordinates are a, b, c, and makes angles of 60° and 45° with the planes of yz and xz respectively.

2. Explain what surfaces may be represented by the equation

$$ax^2+by^2+cz^2+2a'yz+2b'xz+2c'xy+2a''x+2b''y+2c''z+d=0,$$

when the coefficients satisfy the condition

$$abc-aa'^2-bb'^2-cc'^2+2a'b'c'=0,$$

and determine the distinctive characteristics of each.

Ex. $x^2+4y^2-z^2-2yz-xz+4xy+2z=0.$

1842 Find the positions of the umbilici in the surface of which the equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$.

1846 1. Supposing the condition that two lines in space referred to rectangular co-ordinates are at right angles to each other is satisfied, find the equation to a plane containing one of the lines and perpendicular to the other.

2. Can any general conclusion be drawn as to the number of the lines of curvature at an umbilicus, from the fact that at such points P, Q and R vanish? Illustrate your remarks by reference to the umbilici of an ellipsoid and the vertex of a surface of revolution.

1. Find the equation to the plane which passes through 1847
two given points and is perpendicular to a given plane; and thence deduce the equations to a line passing through a given point and perpendicular to a given plane, by supposing the two given points to coincide.

2. Find the surface of revolution which satisfies the equation $rt - s^2 = 0$ at every point.

3. Assuming this property, that in any conic sections if PQ, PR make equal angles with a fixed chord PK and QR be joined, prove that QR will pass through a fixed point for all positions of PQ, PR; prove that if PQ, PR be any two chords of an ellipsoid, in the same plane with and inclined at the same angle to a fixed chord PK, then the locus of all the possible intersections of QR, $Q'R'$ will be the line PK and a plane curve.

1. From the analytical conditions of a straight line being 1848
perpendicular to a plane, find the length of the perpendicular from the origin upon the plane,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

2. Determine the relations among the constants in the equations

$$\frac{x-a}{\lambda} = \frac{y-\beta}{\mu} = \frac{z-\gamma}{\nu}, \quad Ax + By + Cz = D,$$

in order that they may represent respectively a tangent line and normal plane to the curve whose equations are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0.$$

3. Shew that the sum of the lengths of any system of conjugate diameters in an ellipsoid is greatest when they are equal.

4. Prove that the surface $x^2 + y^2 = a^2 + z^2$ may be generated by the continuous motion of a straight line. Does it follow from this circumstance that the tangent plane always touches the surface along a straight line?

5. Write down the general integral equation, and the partial differential equation to surfaces of revolution when the origin lies in the axis of revolution, and by comparing either with the corresponding general equation of the second degree, determine the conditions which must subsist among the coefficients of the latter in order that it may represent a surface of revolution.

1849 1. From the expression for the area of a triangle in terms of the rectangular coordinates of its angular points deduce the equations of a straight line.

2. Determine the nature and position of the surface represented by $xy + zx + yz = a^2$.

3. Determine what must be the form of a column in order that it may appear to be of uniform thickness to an observer in a given position.

DIFFERENTIAL EQUATIONS.

1838 1. INTEGRATE the equation $2xyd_xz + (x^2 + y^2)\, d_yz = 0$.

2. Determine the relations of y to x from the following equations: $y^n + my^{n-1} \,.\, x \,.\, d_xy = y^m d_xy$,

$(y^2 - x^2)\, d_xy + 2xy = 0$; and $d_x^2 y - y = \epsilon^{mx} \,.\, \cos x$.

1839 1. Is $(x + y)^2 - 4ay = 0$ a particular integral or a singular solution of the equation

$$xd_xy - y = yd_xy + (a - x)\,(d_xy)^2?$$

2. Integrate the following equations,

$$d_xy + \frac{xy}{1 - x^2} = x\sqrt{y}, \quad \text{and} \quad \frac{d_xz}{x} + \frac{d_yz}{y} = 1.$$

1840 1. Integrate the equation $\frac{d^2y}{dx^2} - 4m\frac{dy}{dx} + 4m^2y = \sin nx$.

2. Integrate the following equations:

(1) $(y + z)\, dx + (z - x)\, dy - (x + y)\, dz = 0$.

(2) $ydx - xdy - dz = 0$.

(3) $xzp + yzq = xy$.

(4) $1 + mp^2 + nq^2 = 0$.

1. Integrate the equation $\frac{dy}{dx} + \frac{xy}{x^2 - a^2} = \frac{a^4}{x^4}$. 1841

2. Integrate the following equations:

(1) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = e^x \sin x$, by the method of parameters.

(2) $z(z+y)\,dx + z(z-x)\,dy + y(x+y)\,dz = 0$.

3. Shew that an infinite series of the form

$$u = Ae^{\alpha^2 t} e^{\alpha x} + Be^{\beta^2 t} e^{\beta x} + Ce^{\gamma^2 t} e^{\gamma x} + \&c.$$

is the complete integral of the equation $\frac{du}{dt} = a\frac{d^2u}{dx^2}$.

4. Is $y = x$ a singular solution or a particular integral of

$$(x+a)\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = a?$$

1. Integrate the equations $(1+x^2)\frac{dy}{dx} + xy + 1 = 0$, 1842

$$y^3\frac{d^2y}{dx^2} = a^2.$$

2. Integrate the following equations:

(1) $d_x^2 y + n^2 y = \cos nx$.

(2) $zy\,dx + xz\,dy + (xy + az^2)\,dz = 0$.

(3) $x\,d_{(x)}z + y\,d_{(y)}z = nz^n$.

1. Integrate the equations 1843

$$xy\frac{dy}{dx} + x^2 = y^2, \quad y\left(\frac{dy}{dx}\right)^2 + (2x - a)\frac{dy}{dx} - y = 0.$$

2. Integrate $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = x^2$, and $u_{x+1} = a(u_x)^n$.

If $z = \phi\left(\frac{y}{x}\right)$, and $\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} = 0$, find the form of ϕ.

1. Integrate the equations: 1844

$$y\frac{dy}{dx} = 1 + \frac{dy^2}{dx^2}, \qquad \frac{dz}{dy} = \frac{x}{y}\frac{dz}{dx} + \frac{y}{x}.$$

2. If $\sqrt{a}\cdot\frac{dy}{dx} = \sqrt{y-a}$, is the Differential Equation to a family of curves, the singular solution represents their envelope.

3. Integrate the equation

$$\frac{d^2z}{dx^2} - 2\frac{d^2z}{dx\,dy} + \frac{d^2z}{dy^2} = 0.$$

1845 1. Reduce

$$(ax + by)(x\,dy - y\,dx) + (a'x + b'y)\,dy + (a''x + b''y)\,dx = 0$$

to a linear equation, and integrate the following equations,

$$x\,dy + y\,dx = xyz\,dz, \qquad \frac{dz}{dx}\frac{dz}{dy} = 1.$$

2. Integrate the partial differential equation

$$z - x\frac{dz}{dx} - y\frac{dz}{dy} + xy\frac{d^2z}{dx\,dy} = 0,$$

by transforming it into one in which $\frac{z}{xy}$ is the dependent variable, or by any other method, and determine the arbitrary function so that

$$\frac{1}{a}\frac{dz}{dx} = \frac{1}{b}\frac{dz}{dy} = x \text{ when } y = x, \text{ and } z = \frac{a+b}{2} \text{ when } x = y = 1.$$

3. Integrate the equation:

$$\frac{dy}{dx} + ny\tan nx = n\sec nx - m\cos nx\tan mx.$$

4. The equation of the orthogonal trajectories to a system of similar ellipses having their axes coincident is $\frac{y}{x} = \left(\frac{y}{c}\right)^{e^2}$, where e is the common eccentricity of the ellipses.

1846 Deduce the integral of the equation

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + \phi(y)\left(\frac{dy}{dx}\right)^2 = 0$$

from the integral of the equation which results from the omission of one of the last two terms.

1847 1. Find the differential equation of which $y = c(x + c)^2$ is the complete integral; find also the singular solution.

Integrate the equations

$$(y^2 + yz)\,dx + (z^2 + zx)\,dy + (y^2 - xy)\,dz = 0,$$
$$(y + z)\,dx + (z + x)\,dy = (x + y)\,dz.$$

2. Integrate the equation

$$(1 + x^2)\frac{dz}{dx} + \left(\frac{y}{x} - xy\right)\frac{dz}{dy} + \frac{z}{x - y}\left(1 + x^2 - \frac{y}{x} + xy\right) = 0.$$

3. Find the conditions which are necessary and sufficient

in order that $Pdx + Qdy + Rdz$ may be a perfect differential of a function of three independent variables.

If $\frac{dR}{dy} = \frac{dQ}{dz}$ and $\frac{dP}{dz} = \frac{dR}{dx}$, prove that $\frac{dQ}{dx} - \frac{dP}{dy}$ is independent of z.

1. Integrate the equation 1848

$$(ay + bx + c)\frac{dy}{dx} + a'y + b'x + c' = 0.$$

2. Give a geometrical illustration of Part II. Prop. 2., supposing x, y, z to be the co-ordinates of a surface.

Integrate the equation

$$x\frac{dz}{dx} - y\frac{dz}{dy} = \frac{x+y}{x-y}z.$$

3. Integrate the equation

$$(a+bx)^2\frac{d^2y}{dx^2} + A(a+bx)\frac{dy}{dx} + By = 0,$$

where A and B are constant: and thence deduce the integral of the equation

$$(a+bx)^2\frac{d^2y}{dx^2} + A(a+bx)\frac{dy}{dx} + By = X,$$

X being a function of x.

1. Determine the curve which cuts a series of curves described after a given law, at an angle which is a given function of the co-ordinates of the point of section. 1849

A series of circles touch each other at a common point, find the curve which cuts them each at an angle proportional to the vectorial angle of the point of section, their common point being the pole, and their common diameter the initial radius.

2. Prove that a singular solution of an equation of the form

$$P\frac{dy^2}{dx^2} + Q\frac{dy}{dx} + R = 0,$$

will be such a relation between x and y as shall make the left-hand member of the equation a perfect square; and give a geometrical interpretation of this result.

3. Integrate the equations:

$$(1)\quad \frac{dz}{dx} = \phi(z)\frac{dz}{dy}.$$

4. Integrate the equations

$$\frac{d^2x}{dt^2} + (2m^2 - n^2)\,x + 2\,(n^2 - m^2)\,y = 0,$$

$$\frac{d^2y}{dt^2} + (2n^2 - m^2)\,y + (m^2 - n^2)\,x = 0.$$

DEFINITE INTEGRALS.

1843 FIND the value of $\int_0^{\frac{\pi}{4}} dx\,(\tan x)^n$.

1844 Prove that

$$\{\int_0^\infty e^{-x^{2n}}\,dx\}^2 = \tfrac{1}{2}\int_0^\infty e^{-x^n}\,dx \int_0^\infty \frac{dx}{\sqrt[n]{1+x^{2n}}}.$$

1846 Evaluate the definite integral $\int_0^\infty e^{-a^2x^2}\cos rx\,dx$.

1849 Explain the use of the expression

$$\Sigma u_x = \int u_x dx - \tfrac{1}{2}u_x + \frac{B_1}{1.2}\frac{du_x}{dx} - \frac{B_2}{1.2.3.4}\frac{d^3u_x}{dx^3} + \ldots$$

in the numerical summation of slowly convergent series, taking for example the infinite series

$$1 - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{9} + \tfrac{1}{10} - \ldots$$

Why is it necessary to sum a few terms of such a series directly, before applying the formula?

FINITE DIFFERENCES.

1838 PROVE that $\Sigma^n a^{-x}u_x = a^{n-x}\,(D-a)^{-n}u_x$; and shew that

$$(D-a)^{-n}\,0 = a^x\,\{C_1 + C_2x + \ldots + C_nx^{n-1}\}.$$

Shew also that a particular value of $\Sigma^n a^{-x}\cos\theta x$ is

$$\frac{a^{n-x}\,.\,(D^{-1}-a)^n\cos\theta x}{(1-2a\cos\theta+a^2)^n}.$$

1839 1. Find the sum of (n) terms of the series

$$2\,.\,1^2 + 3\,.\,2^2 + 4\,.\,3^2 + \ldots$$

2. Find Σu_x in terms of $\int_x u_x$, u_x, $d_x u_x$ &c.; and shew how

the numerical value of any coefficient may be determined. Approximate to the value of $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}$ + &c. when n is very large.

3. If ϕt be the generating function of u_x, find the generating function of $d_x^r u_{x+r\Delta}$; and thence shew that

$$u_{x+m\Delta} = u_x + m\,(m-n)^0\, d_x u_{x+n\Delta} \cdot \frac{h}{1} + m\,.\,(m-2n)^1\, d_x^2 u_{x+2n\Delta} \frac{h^2}{1.2} + \ldots$$

$$+ m\,.\,(m-rn)^{r-1}\, d_x^r u_{x+rn\Delta} \frac{h^r}{1.2\ldots r} + \ldots$$

1. Find the sum of n terms of the series 1840

$$1\,.\,3^2 + 3\,.\,5^2 + 5\,.\,7^2 + \ldots$$

2. Integrate the equation $u_{x+3} - 3u_{x+1} - 2u_x = c\alpha^x$.

1. Sum the series $1^2 - 2^2 + 3^2 - 4^2$ + &c. to n terms, and 1841 shew that

$$x^n = (-1)^n \left\{\Delta o^{n+1} - \frac{\Delta^2 o^{n+1}}{1.2}(x+1) + \frac{\Delta^3 o^{n+1}}{1.2.3}(x+1)(x+2) - \&c.\right\}$$

2. Find all the solutions of the equation $(\Delta u_x)^2 - a^2 = 0$.

1. Integrate the equations $u_{x+3} - 3u_{x+1} - 2u_x = x^2 + c\epsilon^x$. 1842

$$(u_{x+1})^2 = a\,(u_x)^3. \qquad u_x u_{x+1} - 2u_x + 1 = 0.$$

2. Shew that $\Delta^n 0^x$, $\Delta^n 0^{n+1}$, $\Delta^n 0^{n+2}$... form a recurring series.

Shew that the sum of the series 1843

$$\frac{1}{\sin\theta} + \frac{1}{\sin 2\theta} + \frac{1}{\sin 4\theta} + \ldots \text{ to } x \text{ terms},$$

is equal to $\cot\frac{\theta}{2} - \cot 2^{x-1}\theta$.

Integrate the equation $u_{x+1} - x^2 u_x = (1\,.\,2\,.\,3\,.\ldots.x)^2$. 1844

Prove that $y = ax + a^2$ and $y = \left\{\frac{1}{4} - b\,(-1)^x\right\}^2 - \frac{x^2}{4}$ are distinct 1845 solutions of the same equation in finite differences; and hence shew how either of them may be derived from the other by treating the parameter (a or b) as variable.

Find the equation to a curve such that if the spiral angles 1846 corresponding to any three points be in an arithmetical pro-

gression, the common difference of which is given, the mean radius vector shall be equal to the sum of the extremes.

1847 1. Find $\Delta \frac{1}{u_x u_{x+1} \dots u_{x+n-1}}$, where $u_x = a + bx$, and apply the result to sum the following series to n terms:

$$\frac{1}{2.4.5} + \frac{3}{3.5.6} + \frac{5}{4.6.7} + \dots$$

2. Given $u_x = 2u_{x-1} + 3$ when x is even
and $u_x = 3u_{x-1} + 2$ when x is odd;
find the general expression for u.

3. Is there any essential distinction between those integrals of an equation of finite differences which have been called direct and indirect? Deduce *both* integrals of the equation $y = x\Delta y + (1 - \Delta y)^2$ from the equation obtained by taking its first difference.

4. Shew that $\Delta \frac{\log 2 \sin 2^x\theta}{2^x} = -\frac{\log \tan 2^x\theta}{2^{x+1}}$,

and thence prove that

$$(\tan 2\theta)^{\frac{1}{2}} \tan 2^2\theta)^{\frac{1}{4}} (\tan 2^3\theta)^{\frac{1}{8}} \dots\dots (\tan 2^x\theta)^{\frac{1}{2^x}} = \frac{2 \sin 2\theta}{(2 \sin 2^{x+1}\theta)^{\frac{1}{2^x}}}.$$

1849 1. Find the sum of the first n terms of the series

$$1^3 - 3^3 + 5^3 - 7^3 + \dots$$

2. From the theorem

$$f(e^t) = f(1 + \Delta)\, 1 + f(1 + \Delta)\, 0 \,.\, t + \&c.$$
$$+ \frac{1}{\lfloor m} \,.\, f(1 + \Delta)\, 0^m \,.\, t^m + \&c.$$

where $f(e^t)$ can be expanded in a series of ascending powers of t prove the following expression for Bernoulli's numbers,

$$B_{2n-1} = (-1)^{n+1} \left\{ -\tfrac{1}{2} \,.\, \Delta 0^{2n} + \tfrac{1}{3} \,.\, \Delta^2 0^{2n} - \&c. \quad + \frac{1}{2n+1} \,.\, \Delta^{2n} 0^{2n} \right\}.$$

CALCULUS OF VARIATIONS.

1839 1. WHAT must be the nature of the base of a right cylinder, that when it moves through a fluid in a direction perpendicular to the axis, the resistance, (on the usual suppositions) may be the least possible.

2. Find the curve down which a body will fall in the shortest time possible when acted upon at any point (x, y) by forces X, Y which are any given functions of x and y: solve the problem also when X and Y are such that $d_x Y = d_y X$.

Find $\delta \int_x V$ where V is given by a differential equation 1840
$d_x V + U = 0$; U involving $V, x, y, p, q,$ &c., and apply the result to determine the curve down which a body falling between two given curves, in a medium whose resistance varies as the square of the velocity, shall acquire the greatest velocity; and shew that the tangents at the points of intersection of the curve with the two limiting curves are parallel to each other.

1. From the variation of $\int_x F\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}, \text{&c.}\right)$ subject 1841
to the condition that $f\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}, \text{&c.}\right) = 0$, determine the curve of quickest descent in a medium resisting as any function of the velocity.

2. Find the relation between x and y in order that

$$\int \left(ax^2 \frac{d^2y}{dx^2} - 2bxy \frac{dy}{dx} + cy^2\right) dx$$

may be a maximum or minimum.

Find the curve which generates, by its revolution round the 1842
axis of x, the solid of least surface when the area is given.

Between two fixed points in space draw a curve of given 1843
length, such that its projection on a given plane may subtend at a given point in the plane a sector of maximum area.

1. Shew that the shortest curve joining two points on a 1844
sphere is the arc of the great circle passing through them.

2. Find the relation between x and y which will make $f(x^2 + y^2 - fy\,dx)\,dx$ a minimum.

From the variation of $\int f\left(y, \frac{dy}{dx}\right) dx$ the limits being invari- 1846
able, prove that the time of a body's descent in a vertical plane from one given point to another by a curve, which is convex

upwards, is shorter than that by any other convex curve which includes the former.

1848 If $u = \int_{x_1}^{x_2} V dx$, where V contains $x, y, \frac{dy}{dx} \cdots \frac{d^n y}{dx^n}$, shew that the integral equation which must subsist between x and y in order that u may be a maximum or minimum will generally contain $2n$ arbitrary constants.

Point out how these constants are to be determined, (1) when the values of the quantities $x, y, \frac{dy}{dx} \cdots \frac{d^n y}{dx^n}$ are given at the limits; (2) when these values are only connected by a given system of equations.

1849 Determine the differential equations of the shortest line between two given points upon a curve surface; and prove that the osculating plane at any point of the line contains the normal to the surface at that point.

CALCULUS OF FUNCTIONS.

1841 If $z = \phi(ax - y) + \psi(\beta x - y)$, required the forms of ϕ and ψ, so that when $y = Ax$, $z = Bx$ and when $y = ax$, $z = bx$.

ELEMENTARY STATICS.

1838 A FORCE P acting horizontally supports a weight W on a given inclined plane.

Examine the conditions of equilibrium supposing the plane rough, and friction equal to $\left(\frac{1}{n}\right)^{\text{th}}$ of the pressure.

1839 If G be the centre of gravity of a triangle ABC, three forces in the direction of and proportional to GA, GB, GC will keep the point at rest.

1840 1. Find the centre of gravity of the figure formed by cutting off one corner from a square by a line which bisects the sides.

2. A hemisphere is supported by friction with its curved surface resting upon a horizontal and in contact with a vertical plane; find the limiting position of equilibrium.

3. Explain how a vessel is enabled to sail in a direction nearly opposite to that of the wind.

A door rests on two hinges in a vertical line; determine the lateral pressure on the hinges, and shew that the portion of the vertical pressure supported by each is indeterminate. 1842

A solid sector of a sphere hangs from a point in its circular rim with its axis horizontal, find its vertical angle. 1843

1. Find the centre of gravity, 1844

(1) Of the area of a portion of a parabola bounded by the axis and semi-latus rectum,

(2) Of the surface of a given sector of a sphere.

2. Apply the principle of virtual velocities to find the equation of the surface on which a particle, acted on by any number of centres of force, each varying as any power of the distance, will rest in all positions.

If at the highest point of the path of projectile the velocity be altered without changing the direction of motion, will the time of reaching the horizontal plane which passes through the point of projection be altered? Shew that if the alteration be such that the new velocity is equal to the product of the old velocity and the tangent of the angle at which the body was originally projected, the focus of the new path will be in that horizontal plane. 1846

1. If the angle between two equal forces acting on a point be 120°, what is their resultant? 1847

2. Find the centre of gravity of five equal heavy particles placed at five of the angular points of a regular hexagon.

1. *If the roughness of a plane, which is inclined to the horizon at a known angle, be such that a body will just rest supported on it, find the least force along the plane requisite to drag the body up. 1848

2. *Explain how the force of the current may be taken advantage of to urge a ferry-boat across a river, the centre of the boat being attached, by means of a long rope, to a mooring in the middle of the stream.

3. *AB is a rod capable of turning freely about its extremity A, which is fixed, CD is another rod equal to $2AB$ and attached at its middle point to the extremity B of the former, so as to turn freely about this point; a given force acts at C in the direction of CA, find the force which must be applied at D in order to produce equilibrium.

4. *If a set of forces, acting at the angular points of a plane polygon, be represented by the sides, taken in order, shew that their tendency to turn a body about an axis perpendicular to the plane of the polygon is the same, through whatever point of the plane the axis passes.

5. *Shew that the centre of gravity of a triangular area coincides with that of three particles of equal weight placed at the angular points, and thence deduce its position.

6. Explain how a person sitting in a small carriage can move himself onwards by exerting a pressure on a crank attached to the wheel.

1849 1. *The resultant of two forces is 10 lbs., one of them is equal to 8 lbs., and the direction of the other is inclined to the resultant at an angle of 36°. Find the other force, and the angle between the two.

2. *Two forces act in the same plane at given points in a rigid body; if any point of a certain straight line be fixed, shew that there will be equilibrium. Also if perpendiculars be drawn from this point upon the directions of the forces, the forces will be to each other inversely as these perpendiculars.

3. *A triangular board of given weight rests in equilibrium with its base on a horizontal plane sufficiently rough to prevent all sliding. A force acts upon it in its own plane and in a given line drawn through the vertex and without the triangle;

find by a geometrical construction, or otherwise, the limits between which the magnitude of the force must lie if the equilibrium is preserved.

4. *A triangular plane ABC is kept in equilibrium by three systems of pullies of the above kind, each having one block fastened to a fixed external point and the other attached to an angular point of the triangle by a string whose direction bisects the angle. The same string passes round all the pullies and is solicited by a certain force. Shew that the numbers of the strings between the pullies are as $\cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}$.

5. *One corner of a triangle, equal to $\frac{1}{n}$th part of its area, is cut off by a line parallel to its base; find the centre of gravity of the remainder.

6. *Two equal circular disks with smooth edges, placed on their flat sides in the corner between two smooth vertical planes inclined at a given angle, touch each other in the line bisecting the angle. Find the radius of the least disk which may be pressed between them without causing them to separate.

ANALYTICAL STATICS.

1. FORCES 1,—2,—4, 5 act upon a rigid rod at distances from one end of it proportional to their respective magnitudes, determine the resultant. 1838

2. Apply Guldin's Theorem to find the surface of the solid formed by the revolution of a cycloid round its base.

1. A prolate spheroid rests with its smaller end upon a horizontal plane, find whether the equilibrium is stable or unstable. 1840

2. Find the locus of the centres of gravity of all sections of a given paraboloid, made by planes which are parallel to the axis and to each other.

1843 A cube is acted on by four forces, one in a diagonal, the others in edges, no two of which are in the same plane, and which do not meet the diagonal, find the resultant.

1844 Find the centre of a system of four equal bodies, whose centres of gravity are at the corners of a triangular pyramid.

1845 1. A uniform beam is hung from a fixed point by two unequal strings attached to its extremities; compare the tension of each string with the weight of the beam.

2. W being supported on a smooth inclined plane by a string parallel to the plane, which passes over a fixed pulley, and is attached to a weight W', prove that when W' is moved vertically the centre of gravity of W and W' will neither rise nor fall.

3. A point in a rigid body is fixed, if the forces acting on it are not in equilibrium, find the plane in which a couple must be applied in order to produce equilibrium.

1846 1. Shew that a system of forces, acting in one plane along the sides of a polygon taken in order, and represented in magnitude by the sides in which they act, is equivalent to a couple the moment of which is represented by twice the area of the polygon.

2. Find the tension at any point of a common catenary. If the string be slightly extensible, shew that its whole extension will be proportional to the product of its length and the height of its centre of gravity above the directrix.

1848 1. Apply the equations of virtual velocities to Roberval's balance, and find the horizontal reaction at one of the hinges supporting the piece to which the weight is attached.

2. If a rigid body acted on by a couple receive an indefinitely small arbitrary displacement, shew that the sum of the two forces multiplied respectively by their virtual velocities is constant, provided the moment of the couple and the direction of its plane be given.

3. A chain of given length and weight, and loaded with a given weight at a given point, is suspended between two given points; shew how to determine all the arbitrary constants which enter into the problem of finding the position of equilibrium of the chain, and the tension at any point.

4. A figure of revolution rests with its vertex on a rough horizontal plane; what is meant by the equilibrium being neutral? In this case how may we ascertain whether the body will return to its original position after a small disturbance?

5. An elastic string of variable thickness is extended by a given force; find the whole extension, and determine the quantity of labouring force stored up in a given element of the string, assuming the usual law of elasticity.

6. If two scales, one containing a weight P and the other a weight Q, be suspended by a string over a rough sphere, and if Q be on the point of descending, then the weight $\frac{Q^2-P^2}{P}$ put into the opposite scale will make that scale to be on the point of descending.

ELEMENTARY DYNAMICS.

1. If a heavy body fall from rest through 144 feet, de- 1838
termine the time of motion.

2. A heavy body is projected perpendicularly upwards with a velocity of 25 feet, determine its height and velocity at the end of 2 seconds.

1. A pendulum oscillates in a cycloid, find the number of 1839
such oscillations in a given time, and how that number will be affected by slight contemporaneous changes in the length of the pendulum, and the force of gravity.

2. Having given the velocities at two points of the path of a projectile, find the difference of their altitudes above a horizontal plane.

1840 1. A body slides from rest down a smooth slooping roof, and then falls to the ground; find

(1) the time of sliding down the roof, and the velocity acquired;

(2) the time of falling, and the point where it reaches the ground.

2. Assuming that the friction upon a railroad is $\frac{1}{200}$th of the weight, and that on a level part the wheels of the engine-carriage will not slide, so long as the power requisite for drawing the train does not exceed $\frac{1}{20}$th of the weight of the engine, determine the greatest inclination up which the train can move.

1841 1. There are two weights on a straight lever, neglecting the weight of the lever. If the lever descend from a horizontal position, find the velocities of the weights when it reaches the vertical position.

2. A heavy body is projected perpendicularly upwards with a velocity of 96 feet; find its height and velocity at the end of four seconds.

1842 If the mass of one of them be to that of the other as their common elasticity to unity, shew that the motion of the first after impact will be parallel to the plane touching the spheres at the point of contact, if the motion of the other ball were so previously.

1843 1. Determine the directions in which a particle must be projected with a given velocity, in order that it may hit a mark at a given distance on the same horizontal line with the point of projection.

2. State the principles on which is founded the theory of the direct impact of elastic bodies. Two equal and imperfectly elastic bodies impinge directly; having given the original velocity of the one, find that of the other, in order that after the blow it may be reduced to rest.

1844 1. A body is projected with a vertical velocity (16.7), and a horizontal velocity (.8), prove that its distance from the

point of projection at the end of one second is one foot ($g = 32.2$ feet).

2. An inelastic body impinges upon another of twice its mass at rest, prove that it loses two-thirds of its velocity by the impact.

1. A body is thrown vertically upwards with a velocity $3g$. 1845
At what times will its height be $4g$, and what will be its velocity at these times?

2. A perfectly elastic ball impinges upon another at rest; if the original direction of the striking ball is inclined at an angle of 45° to the line joining the centres, what will be the angle between the directions of its motion before and after impact?

3. Determine the angle of elevation at which a body must be projected in order that the focus of its path may lie in the horizontal plane passing through the point of projection.

1. Three perfectly elastic balls, the masses of which are 1846
A, B and C respectively, are placed with their centres in a straight line; A is made to impinge with a given velocity on B, and B consequently on C; compare the velocity thus communicated to C with the velocity which would have been communicated if the ball B had not been interposed.

2. A body is projected vertically upwards with a velocity which will carry it to a height of $2g$ feet; after how long a time will it be descending with a velocity g?

3. A heavy particle is projected in a horizontal direction with a given velocity; find the latus rectum of the parabola described.

1. Shew that the equations which we employ in Analytical 1847
Geometry, Dynamics, &c. must be homogeneous with reference to each of the independent units to which the concrete quantities which enter into them are referred.

What independent units are employed in Dynamics? Apply the above principle to shew that the equation $t = \frac{\pi\mu}{a^3} + \left(\frac{v}{a}\right)^{\frac{1}{2}}$

must be erroneous, where t is a time, $\frac{\mu}{a^2}$ a central force, a a line, and v a velocity, provided no concrete quantities in this equation are expressed by their numerical values.

2. If the numerical value of the accelerating force of gravity be g when a second is taken for the unit of time, what is its numerical value when half a second is taken for the unit?

3. If any number of bodies be projected in different directions from the same point with equal velocities, shew that the foci of the parabolas described will lie in the surface of a sphere.

1848 1. If the first, A, of three perfectly elastic balls placed in a line impinge directly with a given velocity on the second B, so that B in turn impinges on the third C, find the mass of B in order that the velocity given to C may be the greatest possible, the masses of A and C being known.

2. A train of connected bodies, whose weights are W_1, W_2,..., are moving together in a straight line, being acted on by the retarding pressures P_1, P_2,...respectively, find the conditions in order that the bodies may continue to move with equal velocities when the connexion between them is severed.

1849 1. A particle moves over 7 feet in the first second of the time during which it is observed, and over 11 and 17 feet in the third and sixth seconds respectively. Is this consistent with the supposition of its being subject to the action of a uniform force?

2. If a body be projected with a given velocity so as to pass through a given point, construct the direction of the projection.

3. A heavy particle slides down an inclined plane of given height, if at the bottom of the plane it rebound from a hard horizontal plane, what must be the inclination of the former that the range on the latter may be the greatest possible.

4. One heavy particle draws another up an inclined plane,

by a string passing over a pulley at the top of the plane, state what would take place if the string were cut at any instant.

5. A pendulum which oscillates seconds at one place is carried to a place where it gains two minutes a day; compare the force of gravity at the latter place with that at the former.

NEWTON. 1, 2, 3.

A BODY is projected from a given point so as to describe an 1842
ellipse round a centre of force varying inversely as the square of the distance; find the position of the major axis, and determine the relation between the direction and the velocity of projection, in order that the major axis of the ellipse may be perpendicular to the initial distance of the body from the centre of force.

1. If the superior planets always appeared stationary in 1845
geocentric opposition, what would be the law of force, supposing the orbits to be circles in the plane of the ecliptic?

2. A body is retained in a conic section by a force directed to the focus; shew that the velocity at the greatest or least distance is to the velocity in a circle at that distance, as the square root of the latus rectum is to the square root of twice the greatest or least distance.

1. What does Newton Book. I. Prop. 1. become when 1846
the centre of force is supposed to move off to an infinite distance?

2. A number of particles are projected from points in a horizontal plane with velocities varying as the cosecants of the angles of projection; shew that the areas of the parabolas described are proportional to the cotangents of those angles.

At what point is the velocity of a body which describes 1847
an ellipse about a force in the focus equal to that in a circle at the same distance?

1. * Draw a figure respresenting an intermediate state of 1848
the construction in Lemma VII., when the point B has moved up into a position nearer to A.

2. * How much must the length of the day be shortened, in order that the rotation of the Earth may be sufficiently rapid to destroy the weight of bodies at the equator?

3. * If the latus rectum of a parabola is 24 feet, and the velocity of a body revolving in it at the vertex is 2 yards per minute, find the time in which the body moves from the vertex to one end of the latus rectum.

1849 1. * In Lemma X, if the velocity vary as the square of the time, shew that the space will vary as the cube of the time.

2. * Describe an instrument which is adapted for drawing curves similar to given curves.

DYNAMICS PARTICLE.

1839 1. DETERMINE the point at which a heavy material particle falling down the convex surface of a circular arc, will leave the curve.

2. Investigate the equations of motion of a material point urged by any accelerating forces; and shew that, if the only force acting upon the point be parallel to the axis of (z), its intensity will be equal to

$$a^2 d_{(x)}{}^2 z + 2ab d_{(x)} d_{(y)} z + b^2 d_{(y)}{}^2 z,$$

where (a) and (b) are the constant velocities parallel to the axes of x and y respectively.

1840 1. Express the time of oscillation of a pendulum in a circular arc by means of an elliptic function, and shew that if $2a$ be the arc of vibration, the time

$$= 2\sqrt{\frac{l}{g}} \sec^2 \frac{a}{4} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \tan^4 \frac{a}{4} \sin^2 \phi}}.$$

2. Investigate the equation to the curve, from every point of which a body acted upon by gravity will descend to a given point in half a second of time.

3. A body is projected in a medium whose resistance = k (velocity)^n, and is acted on by no extraneous force; find the

whole space described and the whole time of motion, for all values of n.

A particle moves in a smooth rectilineal tube, which revolves 1842
in a horizontal plane about a vertical axis, and the path of the particle is a circle which passes through the vertical axis: find the angular velocity of the tube in any given position.

1. A body attached to a fixed point by a string revolves in 1846
a vertical circle; find the arc described in a quarter of the time of one revolution after the body leaves the highest point, the tension of the string at that point being zero.

2. Find the velocity of a body which describes an ellipse about a centre of force in the focus; and shew that if the body be projected with this velocity along the radius vector produced, it will recede to a distance from the centre of force equal to the axis major.

3. A body acted on by gravity is projected in a given direction in a medium of which the resistance varies as the velocity; from the equation to the path, deduce that to the path 'in vacuo.'

4. A heavy particle runs down from the highest point of the surface of an attracting sphere having the Earth's mean density; determine the diameter of the sphere in order that the arc traced out by the particle before it leaves the sphere may be 1″ greater than it would have been if the sphere had possessed no attractive power.

5. A body describes the spiral, whose equation is $p = r \cdot \sin a$, find the law of force to the pole, from this property, velocity at any point is that due to falling through a quarter of the radius of curvature under the action of the force at that point continued constant.

If a particle move on the interior of a circle, find the 1847
pressure at the lowest point when the velocity is just sufficient to carry the particle entirely round. If r and θ be the polar co-ordinates of a particle moving in a plane under the action of

any forces $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$ is the force in the radius vector and $\frac{1}{r} \cdot \frac{d}{dt} \cdot \left(r^2 \frac{d\theta}{dt}\right)$ is the force perpendicular to it.

Apply these equations to find what must be the law of a force acting perpendicularly to the radius vector, in order that the angular velocity of the particle may be constant. Find also the curve described.

1848 1. Find the force tending to the pole of an equiangular spiral.

2. Supposing the resistance experienced by a sphere descending in a fluid to be expressed by $hv + kv^2$, determine the constants h, k, for a given sphere, by means of the terminal velocities of the sphere itself and of an equal sphere of different weight, but having the same kind of surface.

3. Two particles attracting one another with forces varying as $\frac{\text{mass}}{(\text{dist.})^2}$ are projected perpendicularly to the line which joins them; find the latus rectum of the orbit which one describes relatively to the other.

4. Find the velocity with which a bullet leaves an air-gun, supposing a given length of the barrel occupied by condensed air at the commencement of the motion, neglecting friction and the inertia and resistance of the air. What length of barrel would give the greatest velocity?

1849 1. If the parallelogram of forces be assumed, shew that the second law of motion need not be enunciated so as to include more than one force. If the law be enunciated so as to include several forces, how may the parallelogram of forces be thence proved?

2. If the Earth's velocity were suddenly destroyed, how many days would the Earth take in falling to the Sun?

3. Form the equations of motion of a particle referred to axes which are fixed in the Earth and revolve with it; and thence shew that the motion of a particle (a projectile for ex-

ample) relatively to the Earth's surface is not the same as it would be if the Earth were at rest, and its attraction were compounded with a force equal to the centrifugal force.

DYNAMICS SYSTEM.

1. A HEAVY uniform rod acted on by gravity revolves round an axis at one extremity from a horizontal position, determine the greatest angular velocity it acquires. 1838

2. If two clocks one of which has a second's pendulum and the other a half second's pendulum of the same material be exposed to the same variation of temperature, will the rates of the two clocks be equally affected?

1. Compare the time of a cylinder's descent down an inclined plane when the plane is perfectly smooth, with that when there is sufficient friction to prevent sliding. 1839

2. Shew that the force on the Sun and the motion of the Sun are nearly the same as if a mass, equal to the sum of the masses of the Earth and Moon, were collected at the centre of gravity of the Earth and Moon.

3. If a solid sphere revolving about a diameter expand so as always to remain spherical and of uniform density, determine the change in its angular velocity corresponding to a given change in its magnitude, the mass remaining the same.

1. A rectangular parallelopiped whose base is a square, oscillates about one of its edges which is supported at its extremities in a horizontal position; find the pressures at the points of support, when the body is in a given position, the centre of gravity, at the beginning of the motion being supposed in the horizontal plane which passes through the axis of rotation. 1840

2. Apply the principle of least action, to find the path of a projectile, and that of a body acted upon by no forces and constrained to move on a given surface.

1841 1. A semi-ellipse revolves about a given diameter, which is horizontal, as an axis; find the pressure upon the axis, the initial position being horizontal.

2. If a right cone slide between a horizontal and vertical plane, its axis being always in a vertical plane, find the whole *vis viva* in any position in terms of the angular velocity.

3. A straight rigid rod in motion is acted upon at its extremities by two given constant forces, whose directions are parallel to two fixed lines at right angles to each other, in the plane of the motion; find the angular velocity and the equation to the path described by the centre of gravity.

1842 1. Find the moment of inertia of a cylinder round a tangent to one of its circular ends.

2. Find the centre of spontaneous rotation of an uniform rod when struck at any point.

3. Explain the object and advantages of rifling the barrel of a gun.

1843 1. An uniform beam is supported on two props at its extremities: if one of the props be suddenly removed, find the instantaneous pressure on the other.

2. Find the principal axes passing through the right angle of a right-angled triangle, and the moments of inertia round them.

1844 1. Find the moment of inertia of an elliptical board about its major axis, and about a tangent parallel to its minor axis.

2. A rod hangs with one extremity fixed to a horizontal axis; what angular velocity must be given to it in order that it may just reach its highest position?

3. A ball impinges directly on a beam, at a distance h from a fixed axis, round which the beam turns; find the angular velocity communicated to the beam.

4. Determine the time of rotation of the conical pendulum, supposing it to be composed simply of two homogeneous rods,

jointed at the top of the vertical axis. What use is made of the conical pendulum in the steam engine?

5. If V be the whole *vis viva* of any system of bodies, B, B_1, B_2, &c., acted upon by gravity only, shew that

$$dV = 2(m - M)g\,dz,$$

where z is the vertical co-ordinate of the centre of gravity of any one of the bodies (B), m the mass of B, and M the mass of that body which, put in place of B, would, with the other bodies B_1, B_2, &c. satisfy the conditions of equilibrium.

Apply this formula to solve the following problem:

A mass m raises another mass m_1 by means of a wheel and axle (m acting upon the wheel); the diameter and moment of inertia of the wheel being given, to determine the axle so that m_1 may ascend through a given space in the least possible time (the inertia of the axle may be neglected).

A heavy beam moves about a horizontal axis passing through 1845
one extremity; apply the principle that the motion of rotation is independent of the motion of translation to determine, for any position of the beam, the pressure on the axis in the direction perpendicular to the beam.

1. If the principle of *Vis Viva* apply in the case of a rigid 1846
body oscillating in a manner that all points in it move parallel to a vertical plane, and if θ is the inclination to the horizon of a fixed line in the body parallel to the plane of motion, $\phi(\theta)$ the height of the centre of gravity (G) above a horizontal plane, shew that the length of the simple isochronous pendulum is $\frac{k^2 + SG^2}{\phi''(a)}$, where a is the value of θ, and S the centre of instantaneous rotation.

2. A pendulum is constructed of a horizontal bar attached to the extremities of two equal bars, and these are suspended at two points in a horizontal line about which the oscillation takes place; find the inclination of the bars, in order that a slight alteration of the temperatue may not affect the time of a small oscillation, neglecting the effect of temperature on the position of the points of support.

3. A body is in motion, its instantaneous angular velocities about the principal axes through the centre of gravity are $\omega, \omega', \omega''$, and α, β, γ the angles which those axes make with the direction of motion of the centre of gravity; shew that the body cannot be brought to rest by a single blow, unless

$$A\omega \cos\alpha + B\omega' \cos\beta + C\omega'' \cos\gamma = 0,$$

where A, B, C are the principal moments with respect to the centre of gravity.

1847 1. Find the centre of oscillation of a sphere attached to a fine wire, supposing the wire and the sphere to form a single rigid body; and show by general reasoning that the error introduced by this supposition is very small, provided the length of the string bear a large ratio to the radius of the sphere.

2. A pendulum performs small oscillations, and the arc of oscillation slowly diminishes in geometric progression, on account of friction: assuming the approximate expression for the time of oscillation

$$T = \pi\sqrt{\frac{l}{g}}\left(1 + \frac{a^2}{16}\right),$$

prove the following formula for the reduction of the number N of oscillations observed to the number N_1 of indefinitely small oscillations,

$$N_1 = N\left\{1 + \frac{\sin(A+a)\sin(A-a)}{32\log_e 10\,.\,(\log\sin A - \log\sin a)}\right\},$$

where A, a are the semi-axes of oscillation at the beginning and end of the observation.

1848 1. Define moment of inertia; and find the moment of inertia of a right cone on an elliptic base about the major axis of the ellipse.

2. If a body, supposed to be revolving nearly about the principal axis (x_1) of greatest or least moment, be acted on by a couple $= c\int \omega_2 dt$ in the plane $x_1 z_1$, shew to a first approximation that the motion will be stable provided that

$$\omega_1^2 > \frac{cC}{(A-B)(A-C)}.$$

3. If a body whose principal moments about the centre of gravity are equal be revolving about an axis passing through the centre of gravity, and be constantly acted on by a couple tending to turn the body about an axis at right angles to the instantaneous axis of rotation, what will be the nature of the motion of this latter axis, (1) in space, (2) within the body?

How would the result be modified if the principal moments about the centre of gravity were not equal?

4. State the theoretical grounds on which we might conclude that the combined action of the Sun and Moon on the protuberant matter of the Earth due to its oblateness would not tend to alter permanently the position of the axis of rotation with reference to the Earth.

5. If a rigid body, previously at rest, be set in motion by a single blow, prove that after moving for any time it will be again reduced to rest by an equal blow acting in the same line in an opposite direction.

A cube revolves about one edge which is fixed at a given 1849
inclination to the vertical, find the pressure on the axis, supposing the cube to start from rest at its position of unstable equilibrium.

HYDROSTATICS.

1. A CUBE, each edge of which is 4 inches long weighs 1838
16247 grains in air, and 95 grains in water, determine the weight of a cubic inch of water, having given that the specific gravity of water = 770 × specific gravity of air.

2. A vessel in the form of a tetrahedron with one face horizontal is filled with water, compare the pressures on the sides of the tetrahedron with the weight of the water.

3. If the apparent variation of a common barometer be $\frac{1}{5}$ an inch, and the area of the cistern be 5 times that of the tube, what is the true variation?

4. If an aperture be made in the side of a vessel full of fluid, explain the effect upon the vessel.

1839 1. Upon what property of fluids does the floating of a body depend? Does the magnitude of the part out of the fluid depend entirely upon its specific gravity? If so, account for a body sinking in a quicksand.

2. A right cone, the vertical angle of which is a right angle, floats with its vertex downwards in a fluid whose specific gravity is 8 times as great as that of the cone; determine the nature of the equilibrium.

1840 1. Find the depth to which a given right cone will sink, when floating in a fluid with its axis vertical and base downwards.

2. Account for the phenomenon of dew, and shew that from an observation of the dew-point on a still evening, the minimum temperature of the succeeding night may be approximately predicted.

1841 1. Determine the vertical pressure on the rectangular side of a full trough, inclined at a given acute angle to the horizontal bottom: also, the dimensions of the side being given, find the angle of inclination when this pressure is the greatest.

2. A prism, whose transverse section is a triangle, is just immersed in a fluid with its axis vertical; find the pressures on the sides, and shew that it has no tendency to lateral motion.

3. How is it that the pressure of the atmosphere will support the mercurial barometer, and will not support the water in a glass turned upside down?

4. The specific gravities of three fluids which mix freely are a, b, c; and weights of them are taken in the proportion of $1 : m : n$; find the specific gravity of the compound.

1842 Shew that, when a solid is immersed in a fluid, the weight lost is to the whole weight as the specific gravity of the fluid is to that of the solid. A mineral weighs 65 grains in vacuo and 44 grains in water; what is its specific gravity?

1843 A cylinder with its axis vertical floats in two fluids of different densities, find the ratio of the two parts into which the cylinder is divided by the common surface of the two fluids.

A flexible hollow cone contains elastic fluid. Find the tension at any point of the curve surface. 1844

A cylindrical vessel, the top and bottom of which are formed by planes perpendicular to its axis, contains elastic fluid, the weight of which may be neglected. If the vessel revolve uniformly about its axis, find the pressure at any point of the fluid mass. 1845

1. A fourth part of a sphere, bounded by planes perpendicular to each other and passing through the centre, is immersed in a heavy fluid in such a manner that one of the planes is coincident with the surface of the fluid; find the point at which the resultant of the whole fluid pressure acts. 1846

2. A cylindrical vessel is formed by a flexible and inextensible sheet without weight suspended from two paralled lines in a horizontal plane. Shew that it will settle when filled with incompressible fluid into a surface such that the curvature at any point varies as the depth.

3. Given that a drop of water introduced into a barometer when the temperature is 50° Fahrenheit causes the mercury to fall .37 of an inch, find the volume which a cubic foot of atmospheric air would occupy if deprived of its moisture, the pressure and temperature being unaltered, supposing the barometer to stand at 29.75 inches, and the dew point to be 50°.

1. Define specific gravity; and find the weight of a cubic inch of glass whose specific gravity is 3.456, the weight of a cubic foot of water being a thousand ounces. 1847

2. What is 20° centigrade in Fahrenheit's scale?

3. A cylinder floats in a fluid with its axis horizontal, the specific gravity of the cylinder being half that of the fluid; find the greatest length of the cylinder for which the equilibrium is unstable.

4. When a mass of fluid is in equilibrium, prove that if the portion comprised between the free surface and any other surface of equal pressure be removed, the equilibrium of the

rest will not be affected, provided the forces acting on the fluid do not depend on the mutual attraction of its parts.

Prove that such a portion can also be removed, if the surfaces of equal pressure are similar and concentric ellipsoids, and the particles of the fluid attract each other according to the law of gravitation.

1848 1. * A straight vertical tube is closed at its lower end; how much of a given liquid can be poured into it, the air which originally filled it being compressed at the bottom of the tube?

2. * Does the tension of the rope by which a diving bell is suspended increase or decrease as the bell is sunk lower?

What would be the effect of opening an orifice in the highest point?

1849 1. * The specific gravity of coal is about 1.12, that of water being 1, and a cubic foot of water weighs 1000 oz.; find the edge of a cubical block of coal which weighs 2000 tons.

2. * A cylindrical vessel, the radius of the base of which is 1 foot, contains water; if a cubic foot of cork (sp. gr. = .24) be allowed to float in the water, find the additional pressure sustained by the curved surface, and by the base, respectively.

3. * What must be the vertical angle of a conical vessel, in order that when it is placed with its vertex upwards, and filled with heavy fluid through a hole at the vertex, the pressure on the curved surface may be to the pressure on the base as 4 to 3?

Prove that the ratio above-mentioned cannot for any cone be less than 2 : 3.

4. What must be the vertical angle of a cone which floats with its vertex downwards, in order that the metacentre may be at the centre of the plane of floatation?

5. * In an air pump if instead of the receiver we use a cylindrical vessel of ten times the capacity of the barrel, and

cover the upper extremity with a diaphragm capable of sustaining only half the pressure of the atmosphere, find after how many ascents of the piston the diaphragm will burst.

Given $\log_{10} 2 = 0.3010300$,
$\log_{10} 11 = 1.0413927$.

HYDRODYNAMICS.

A CLOSED cylinder filled with air, is whirled round its vertical axis with a given angular velocity, find the density and pressure throughout any stratum at a given distance from the axis. 1838

Each particle of a fluid attracts with a force which vanishes when the distance of the particle from the attracted point is finite, find the pressure at any point in the interior of the fluid; and determine the equation to the suface of a fluid contiguous to a plate immersed vertically in it. 1839

Find the approximate time of emptying half the fluid from a full conical vessel which has its axis vertical, and a small orifice at the vertex. 1842

Deduce the time of a small oscillation of a fluid in a bent tube of uniform bore in a vertical plane. 1848

1. Shew that the arbitrary constant which occurs in the equation of steady motion is in general only constant along the same line of motion; and find its value in the case of a liquid revolving uniformly about a fixed axis. 1849

2. Apply the Equation of Steady Motion to find the time in which the air condensed in a vessel will be discharged through a small orifice.

GEOMETRICAL OPTICS.

1. IF a ray of light be refracted through a prism rarer than the surrounding medium, in a plane perpendicular to its edge, shew that the deviation is *from* the thicker part of the prism. 1838

2. What is the effect produced upon the image by a par-

ticle of dust on the object-glass of Galileo's telescope, and in what respect does this effect differ from that produced in the astronomical telescope?

1839 1. If a candle be placed upon a table at a given distance from a point in that table, what must be the height of the candle, that the brightness at that point may be the greatest possible?

2. Determine the position of the primary focal line after the n^{th} reflection, when a pencil of parallel rays emerges parallel from a drop of water after $(n+p)$ internal reflexions; and shew how the partial brightness of the sky immediately within the primary rainbow may be explained.

1840 1. At what distance from the eye must a concave lens be placed, that the apparent linear magnitude of a small distant object may be diminished one half?

2. Explain what is meant by saying that objects appear equally bright at all distances, and shew how it is accounted for. Why is the apparent brightness of a star increased by the use of a telescope, whilst that of a planet is not?

3. What are the principal defects in eye-pieces, arising from spherical aberration? Which of them are corrected in the eye-pieces in common use? In Ramsden's eye-piece, two lenses of equal focal length $3l$ are placed at a distance $2l$ from each other, required the forms of the lenses so as to throw the best possible image of a plane object on a distant plane surface, the pencils being defined by a diaphragm placed in the focus of the compound lens.

1841 1. A small diverging pencil is incident directly on a concave reflector; and, a small object being placed at right angles to the axis, half way between the surface and the principal focus, compare the linear magnitudes of the image and object.

2. Find an expression for the indistinctness arising from spherical aberration in the direct passage of a pencil of rays through two lenses; and shew that it is less in a Galileo's telescope than in an astronomical one of the same power.

1. What must be the angle between the two mirrors, in order 1844
that a ray incident parallel to one of them may, after two reflections, be parallel to the other?

2. Shew what kind of lens will be required, to enable a person to see distinctly under water.

3. If an eye be under water, shew how to determine the course of the ray by which any object above the water is seen.

4. Prove that, when a ray of light is reflected at any curve surface, the length of the course of the ray, reckoned from any point in the incident ray to any point in the reflected, is a minimum.

If a polished plane have an indefinite number of very fine concentric circular grooves turned on its surface, and light be incident on it from a luminous point, the appearance presented to the eye of an observer will be that of a bright curve; find its equation.

5. Two astronomical telescopes of given power are placed with their axes in the same straight line, and the object-glass of one toward the eye-glass of the other, find the magnifying power of the combination.

1. Can objects be seen through a prism, whose refractive 1845
index for mean rays is 1 . 5, and refracting angle a right angle?

2. Why is the image of a point formed by a prism in the position of minimum deviation more distinct than when the prism is in any other position?

3. In Newton's telescope with Ramsden's eye-piece, shew what ought to be the form of the plane mirror in order to stop as little light as possible. Supposing the longer diameter of such a mirror 2 inches, and the diameter of the object-mirror 8 inches, find approximately what fraction of each incident pencil is stopped.

1. The focal length of the object-glass of a simple astron- 1846
omical telescope is 15 inches, and its breadth 3 inches; find the focal length of the eye-lens in order that, on looking at a distant

point, the pencil of rays may just fill the pupil of the eye, the breadth of the pupil being one fifth of an inch.

2. Shew that a defect in the eye, of such a nature that the rays of a pencil incident in a horizontal plane are differently refracted from those in a vertical plane, may be remedied by the use of a lens of which one surface is cylindrical and the other spherical. Find the foci of a pencil of parallel rays in the two principal planes, when the radii of the sphere and cylinder are $3\frac{1}{2}$ and $4\frac{1}{2}$ respectively.

1847 If a ray will not pass through when the prism is in the position of minimum deviation, it will not pass through when the prism is in any other position.

1848 1. * Why does a common looking-glass give more than one image of a point?

2. * Explain the appearance which will be presented to an eye placed under water and looking upwards.

3. * Why does a straight rod appear bent when partly immersed in water? What must be its inclination to the horizon when its apparent portions are inclined to each other at the greatest angle?

4. * If the focal lengths of the lenses be 12 inches and 1 inch, how far must the eye-glass be moved for viewing an object at a distance of 40 feet from the object-glass?

5. Find the aberration of a pencil of parallel rays after direct refraction at a spherical surface.

6. Explain how the form of a small oblique pencil after reflection or refraction is represented. Shew how the relative position of the focal lines is connected with the direction of the aberration of a direct pencil.

7. If the minimum deviation of one fixed line be observed, and then the deviations of another line read off without stirring the prism, shew that the angles of incidence and emergence for the second line are immediately determined, and thence obtain, in a form adapted to logarithmic computation, the refractive index of the second line.

8. If the focal length of the object-glass be three feet, and the magnifying power 200, what must be the focal lengths of the lenses which form the eye-piece?

1. * Rays diverging from a point are incident on a plane mirror, within what space must the eye be situated to see a given point by reflexion at the mirror; and within what space must a point be situated to be seen by the eye in a given position? 1849

2. * A pencil of parallel rays is reflected at the surface of a spherical mirror, prove that the intersection of any ray with the axis moves in the direction of the incident light, or in the contrary direction, (according as the mirror is concave or convex,) as the ray considered moves from the axis.

3. * When diverging rays are incident nearly perpendicularly upon a spherical refracting surface, if the conjugate foci are each at a distance from the surface equal to twice the radius, what is the index of refraction?

4. Shew that the fixed lines of the spectrum may be seen distinctly with a prism which is out of the position of minimum deviation, provided the spectrum be in focus for the primary plane. Shew also that in such a position of the prism a defect of parallelism between the edge of the prism and the slit through which the light passes would produce indistinctness.

PLANE ASTRONOMY.

1. The Sun's declinations at noon on the 19th, 20th, 21st of June were respectively 23° . 26′ . 39″, 23° . 27′ . 23″, 23° . 27′ . 42″; find when his declination was 23° . 27′ . 30″. 1838

2. A horizontal dial is marked with hour lines, what is the nature of the correction to be applied to regulate a common watch?

In constructing the dial a small given error is made in the latitude; what correction must be made in the graduation so that the time may be indicated correctly?

1840 Explain the nature and use of the method of least squares.

If x, y, z be quantities given by observation and connected theoretically by the equation $z = a + bx + cy$, determine a, b, c so that the sum of the squares of the differences, between the observed values of z and those which are calculated by this formula from corresponding observations of x and y, may be a minimum.

What is meant by assigning *weights* to the different observations, and how are the resulting values altered by so doing?

1841 If the Sun's declination at three successive noons be $23° . 27' . 31''$, $23° . 27' . 34''$, $23° . 27' . 12''$ respectively, find when the declination is the greatest, and its value.

1843 Shew that in latitude $66° . 32'$ the longest day is 24 hours.

1844 What is the equation of time; and what are the causes of it? Determine whether it is positive or negative in the month of February.

1845 1. Why is the apparent place of the Moon not sensibly affected by aberration?

2. Prove that an error of level in the transit instrument will produce an equal error in the latitude thus determined.

1846 1. In Part II. Prop. 2, Find the correction, in minutes, supposing the height in question to be 16 feet 6 inches, and the radius of the Earth 4000 miles.

2. In drawing a meridian line by means of the shadow of a vertical rod, at what times of the year may the change of the sun's declination be neglected without sensible error?

3. The parallaxes of the Sun and Moon being respectively $8'' .17$ and $57' .03$, find approximately the ratio of the distances of the Sun and Moon from the Earth, and thence shew that the Moon's orbit in space is always concave to the Sun.

4. Investigate a formula for finding the apparent time by means of a horizontal dial which is graduated for a latitude different from that of the place at which it is mounted.

At what times of the year will the right ascension and declination of a given star be respectively unaffected by aberration, and at what hour of the day will a star in the solstitial colure cross the meridian under these circumstances? 1847

1. * The equation of time at noon on one day is $3^m\ 14^s$, and at the succeeding noon is $3^m\ 12^s$, what time ought a correct watch to shew when a sun-dial marks 6 o'clock on the evening of the former day? 1848

2. The Moon's disk not being a complete circle, how may her semidiameter be obtained from observation?

3. From the aberration of a given star in declination determine the coefficient of aberration.

If the plane of a horizontal dial be deflected towards the South through a small given angle find the error in the time indicated by it. 1849

LUNAR THEORY.

1. EXPLAIN fully in what manner, and from the combination of what terms, the inequality called *evection* arises in the process of integrating the equations of motion of the Moon. Shew that this inequality would not arise if the Sun's motion in longitude were neglected. 1843

2. If $\delta\gamma$ and δk be the variations of γ and k introduced by the last term, shew that

$$(\delta\gamma)^2 + \left(\frac{\delta k}{k}\right)^2 = \text{const.}$$

1. Supposing the moon to move in the plane of the ecliptic, and assuming the equations 1844

$$h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) + \frac{du}{d\theta}\frac{dh}{dt} + P = 0,$$

$$u\frac{dh}{dt} - Q = 0,$$

where P is the whole force along the radius vector, and Q that perpendicular to it, h representing $r^2\frac{d\theta}{dt}$; determine the

value of u to the second order of small quantities, so far as it does not depend upon the eccentricity of the lunar orbit.

2. Apply the approximate differential equation to the orbit of the moon, to find a second approximation to the orbit, supposing the moon to move in a resisting medium the resistance of which is k (vel)^2, k being of the same order as the eccentricity, and the sun's disturbing force being neglected.

1847 A further approximation for the moon's longitude gives the term $-\frac{3}{2}m^2n\int_0^t (e'^2 - E'^2)\,dt$ where E' is the value of e' when $t = 0$; what is the effect of this term?

1848 In the Lunar Theory, if we suppose

$$P = \mu u^2, \qquad T = -\frac{3}{2}\frac{m'u'^3}{u}\sin 2(\theta - \theta'),$$

to be the forces on the moon, and $u = a\{1 + e\cos(c\theta - \alpha)\}$ to be a first approximate value of u, find u to the second order.

1849 1. Given to the second order

$$s = k\left\{\sin(g\theta - \gamma) + \frac{3m}{8}\sin[(2 - 2m - g)\theta - 2\beta + \gamma]\right\},$$

$$\frac{S - Ps}{h^2u^3} = \tfrac{3}{2}m^2s\left[1 + \cos\{(2 - 2m)\theta - 2\beta\}\right],$$

$$\frac{T}{h^2u^3} = -\frac{3m^2}{2}\sin\{(2 - 2m)\theta - 2\beta\};$$

find g to the third order.

2. The differential equation for the determination of u in the Lunar Theory involves a term

$$Am^2\frac{a}{a'}\frac{E - M}{E - M}\cos\{(1 - m)\theta - \beta\},$$

where aa' are the mean distances of the Moon and Sun respectively from the Earth, the rest of the notation as usual; find the term introduced into the expression for the longitude, shew that it is of the third order, and explain the use which may be made of it in determining a certain important astronomical element.

PLANETARY THEORY.

If the orbits of two planets which disturb each other, be very nearly circular, shew that the inequalities of the radius vector may be immediately deduced from those of the longitude by means of the equation 1847

$$\frac{\delta r}{r} + \tfrac{1}{2} \cdot \frac{\dfrac{d\delta\theta}{dt}}{\dfrac{d\theta}{dt}} + \tfrac{1}{2} \frac{a}{\mu} \cdot \frac{n}{n - n'} R = 0.$$

1. In the equation $\dfrac{dR}{d\theta} = \dfrac{dR}{d\omega} - \dfrac{dR}{d\epsilon}$, where R is the disturbing function of a planet, and θ is measured in the plane of the body's undisturbed orbit, if $\dfrac{dR}{d\theta} = 0$, what is the direction of the disturbing force? 1848

2. In the development of the disturbing function R of a planet which moves in the ecliptic, what is the practical use of the rule which supplies a limit to the dimensions of the coefficient of a periodic term with a proposed argument. If two planets disturbing one another were revolving in mean periods of 350 and 201 days, what form of terms in their disturbing functions would demand examination?

FIGURE OF THE EARTH.

Shew that if $(1 - 2ph + h^2)^{-\frac{1}{2}} = P_0 + P_1 h + P_2 h^2 + \&c.$ 1844

$$\frac{d}{dp}\left\{(1 - p^2)\frac{dP_n}{dp}\right\} + n(n+1) P_n = 0.$$

Find the figure of the Earth on the hypothesis of its homogeneity; assuming the following expression for the potential of the attraction of a homogeneous body, differing very slightly from a sphere, on an external particle, 1846

$$V = \frac{4\pi\rho a^3}{3r} + \frac{4\pi\rho a a^3}{r}\left(y_0 + \frac{y_1}{3}\frac{a}{r} + \frac{y_2}{5}\frac{a^2}{r^2} + \ldots + \frac{y_n}{2n+1}\frac{a^n}{r^n}\ldots\right).$$

1848 If two arcs, each subtending an angle of one degree, be measured on the meridian, one near the pole, the other near the equator, shew how to deduce the ratio of the axes of the earth.

ATTRACTIONS.

1840 THE attraction of an ellipsoid upon an external point may be made to depend upon the attraction of another ellipsoid upon an internal point. Apply this method to determine the attraction of a sphere upon a point without it, the force varying inversely as the square of the distance.

1848 Find the attraction of an infinite plane on a particle without it; and apply the result to explain the abrupt transition in the resultant attraction of a spherical shell as the attracted particle passes from the outside to the inside.

SOUND.

1840 EXPLAIN the propagation of sound in steam at the greatest density that corresponds to its temperature.

1843 One of the equations, which determine the vibratory motion of the particles of air, may be put in the form

$$\frac{d^2a}{dt^2} = A\left(\frac{d^2a}{dx^2} + \frac{d^2a}{dy^2} + \frac{d^2a}{dz^2}\right).$$

xyz being the co-ordinates (referred to any rectangular axes) of a particle in its equilibrium position, and a the displacement of that particle parallel to the axis of x at any time t during the motion; shew that, if the wave-surfaces be such that a normal at any point of one of them is a normal to all the rest, the value of a is

$$a = \frac{1}{n}\left[f\{\sqrt{(A)}\,.\,t - n\} + \phi\{\sqrt{(A)}\,.\,t + n\}\right],$$

n being that portion of the normal at any point which is a harmonic mean between the greatest and least radii of curvature at that point. Interpret this result.

1847 The same musical note is produced in each of two railway trains travelling in the same direction; shew that a person

standing on the line between the trains will hear beats, and determine the number of beats per second, when the pitch of the note and the velocities of the trains are given.

It is found that water contracts 46 millionths for a pressure 1848
of one atmosphere, or of a column of water of 34 feet nearly; find numerically the velocity of sound in water.

HEAT.

Two parallel indefinite planes are maintained at different 1841
constant temperatures; a thin indefinite plane lamina being placed between them so as to be parallel to each, find the temperature which it will ultimately attain, on the supposition of the space between the planes being a vacuum.

To what causes may the excessive cold of the higher regions of the atmosphere be attributed?

State the steps of the reasoning by which the temperature of that part of space in which the earth moves has been approximated to.

A cylindrical shell of infinite length has its interior surface 1842
kept at a temperature 1 while its exterior surface is kept at the temperature 0: find the permanent temperature at any point.

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INDEX.

	Problems.	*Bookwork.*	*Riders.*
Euclid, ... page	1	139	210
Arithmetic, ...	4		211
Algebra, ...	4	142	213
Theory of Equations, ...	14	146	220
Plane Trigonometry, ...	15	148	222
Spherical Trigonometry, ...	22	151	225
Conic Sections, ... }	25	153	226
Geometry of Two Dimensions, }		155	226
Differential Calculus, ...	39	158	230
Integral Culculus, ...	45	161	234
Geometry of Three Dimensions, ...	47	162	237
Differential Equations, ...	61	164	240
Definite Integrals, ...	64	206	244
Finite Differences, ...	66	206	244
Calculus of Functions, ...	68		248
Calculus of Variations, ...	69	207	246
Statics, Elementary, ...	69	165	248
Statics, Analytical, ...	69	168	251
Dynamics, Elementary, ...	80	169	253
Newton, 1, 2, 3, ...	83	172	257
Dynamics, Particle, ...	85	174	258
Dynamics, Rigid Body, ...	93	177	261
Hydrostatics, ...	105	180	265
Capillary Attractions, ...	111	209	269
Geometrical Optics, ...	114	184	269
Plane Astronomy, ...	122	188	273
Newton's Lunar Theory, &c., ...	128	194	
Lunar Theory, ...	129	195	275
Planetary Theory, ...		197	277
Laplace's Co-efficients, &c., ...	129	199	277
Precession and Nutation, ...	130	201	
Attractions, ...	130	200	278
Physical Optics, ...	133	201	
Sound, Vibrations, ...	135	209	278
Electricity and Magnetism, ...	136	208	
Heat, ...	138	208	279

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